MCW Biostatistics Technical Report 63: BART with logGamma errors using a convolution mixture of Normals

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Bayesian Additive Regression Trees (BART) [1] allows one to regress a continuous outcome, y, on a vector of covariates, $\boldsymbol{x}$, via an arbitrarily flexible function, f, i.e. $y = f(\boldsymbol{x}) + \epsilon$ where $\epsilon \sim \mathsf{N}\left(0,\ \sigma^2\right)$. However, suppose ϵ follows some other distribution function, say $G(\epsilon)$, then we can still employ BART provided that $g(\epsilon)$ can be reliably approximated by a mixture of Normals, i.e. $g(\epsilon) \approx \sum_i p_i \mathsf{N}\left(\mu_i,\ \sigma_i^2\right)$ where (p_i, μ_i, σ_i) are known. We extend BART to the situation where ϵ follows the logGamma distribution with distribution function $G_\alpha(y)$ and density function $g_\alpha(y)$.

If x follows the Gamma distribution, then $y = \log x$ follows the logGamma distribution, i.e. $x \sim \mathsf{Gamma}\left(\alpha,\ 1\right)$ where $\alpha > 0$ implies $y \sim \log \mathsf{Gamma}\left(\alpha,\ 1\right)$ and $g_\alpha(y) = \Gamma(\alpha)^{-1} \mathsf{e}^{y\alpha - \mathsf{e}^y}$.

Fruhwirth-Schnatter, Fruhwirth, Held, and Rue (FFHR) [2] show how to reliably approximate the logGamma distribution, logGamma$\left(\alpha,\ \beta\right)$ with a mixture of Normals. However, their method (which we also refer to as FFHR) is not readily applicable to our work since we need $\alpha < 1$ while FFHR requires that $\alpha \geq 1$. Therefore, we extend FFHR to meet our needs with similar high-degree of accuracy so we can approximate any logGamma$\left(\alpha,\ \beta\right)$ routinely via $\log \beta x_\alpha = \log \beta + y_\alpha$ where $\alpha > 0$ and $\beta > 0$.

FFHR employ the Kullback-Leibler divergence [3] to determine the accuracy of the approximation: $\delta_{KL}(\boldsymbol{\theta}_\alpha) = \int_{-10}^{6} g_\alpha(y) \log \frac{g_\alpha(y)}{\hat{g}_\alpha(y; \boldsymbol{\theta}_\alpha)} \mathrm{d}y$ where $\boldsymbol{\theta}_\alpha = (\boldsymbol{p}_{y_\alpha}, \boldsymbol{\mu}_{y_\alpha}, \boldsymbol{\sigma}_{y_\alpha})$. Then, they use the Nelder-Mead simplex method [4] to minimize the following objective function: $\Delta_{KL}(\boldsymbol{\theta}_\alpha) = \delta_{KL}(\boldsymbol{\theta}_\alpha) + 10^9(1 - \sum_i p_{y_\alpha i})^2$ subject to the constraints $0 < p_{y_\alpha i} < 1$ and $0 < \sigma_{y_\alpha i}$.

We adopt the following notation: y_α is a random variable with the same distribution as $\log x_\alpha$ where $x_\alpha \sim \mathsf{Gamma}\left(\alpha,\ 1\right)$. Notice that we can decompose y_α as $y_\alpha = y_{\alpha+1} + w_\alpha$ where $w_\alpha \sim \mathsf{Exp}\left(\alpha\right)$ and $y_{\alpha+1} \perp w_\alpha$. This result can be extended to what could be called a "distributional factorial" property of the logGamma distribution: $y_\alpha = y_{\alpha+n} + \sum_{i=1}^{n} w_{\alpha+i-1}$ where $w_{\alpha+i-1} \overset{\mathsf{ind}}{\sim} \mathsf{Exp}\left(\alpha + i - 1\right)$ and $y_{\alpha+n} \perp w_{\alpha+i-1}$ for $i = 1, \ldots, n$.

Now, let us return to where our interest resides: $\alpha < 1$. We found it difficult to approximate logGamma for $\alpha < 1$ with $\alpha \geq 1$ in one step via $y_\alpha = y_{\alpha+1} + w_\alpha$. However, the approximation with two steps is satisfactory, i.e. substitute $y_{\alpha+1} = y_{\alpha+2} + w_{\alpha+1}$ into $y_\alpha = y_{\alpha+1} + w_\alpha$ which yields $y_\alpha = y_{\alpha+2} + w_{\alpha+1} + w_\alpha$. We accomplish this by approximating the distribution for each of these 3 terms by their own mixture of Normals the composite of which we call a convolution mixture of Normals.

FFHR provide a high-degree of accuracy for approximation at the integers of $\alpha \in \{1, 2, 3, 4\}$ with mixtures of 10 Normals, i.e. $y_\alpha \overset{\text{approx}}{\sim} \sum_{i=1}^{10} p_{y_\alpha i} \mathsf{N}\left(\mu_{y_\alpha i},\ \sigma^2_{y_\alpha i}\right)$ (note that the FFHR weights, $p_{y_\alpha i}$, are identical, $p_{y_\alpha i} = p_{y_{\alpha'} i}$, for $\alpha \neq \alpha' \in \{1, 2, 3, 4\}$, however, this will not be the case for our approximations for $\alpha \in (2, 3)$). To create mixtures for non-integer $\alpha \in (2, 3)$, we estimate a starting point via linear interpolation starting with $\alpha = 2.5$ as $0.5\hat{\boldsymbol{\theta}}_2 + 0.5\hat{\boldsymbol{\theta}}_3$. Then we plug this starting point into the subplex algorithm [5] where we minimize $\Delta_{KL}(\boldsymbol{\theta}_{2.5})$ arriving at the solution $\hat{\boldsymbol{\theta}}_{2.5} = \arg\min_{\boldsymbol{\theta}_{2.5}} \Delta_{KL}(\boldsymbol{\theta}_{2.5})$. The subplex algorithm (a variant of the Nelder-Mead simplex method) is more efficient and robust than simplex while retaining the latter's facility with discontinuous objectives. Now, we proceed to fill in the grid of 129 points: $2, 2+\frac{1}{128}, \ldots, 2+\frac{127}{128}, 3$; i.e. create a linear interpolation starting point $0.5\hat{\boldsymbol{\theta}}_2 + 0.5\hat{\boldsymbol{\theta}}_{2.5}$ to plug into the subplex method arriving at $\hat{\boldsymbol{\theta}}_{2.25}$, etc. Once you reach this grid level, linear interpolation between the grid points provides sufficiently accurate approximations.

Finally, we approximate the distribution of $w \sim \mathsf{Exp}\,(1)$ by a mixture of 20 Normals. Since the Exponential distribution is restricted to positive values and the Normal distribution is not, we can not employ the Kullback-Leibler divergence. Instead, we rely on integrated squared error: $\delta_{ISE}(\boldsymbol{\zeta}) = \int_0^\infty \left(g_w(y) - \hat{g}_w(y; \boldsymbol{\zeta})\right)^2 \mathrm{d}y$ and use the objective function $\Delta_{ISE}(\boldsymbol{\zeta}) = \delta_{ISE}(\boldsymbol{\zeta}) + 10^9(1 - \sum_i p_{wi})^2$. Accuracy of approximations achieved can be seen visually in the plots of densities (and log-densitites) of $-y,\ y \sim \log \mathsf{Gamma}\,((,\ \alpha)\,, 1)$ for various choices of α in the range of interest on the next several pages.

References

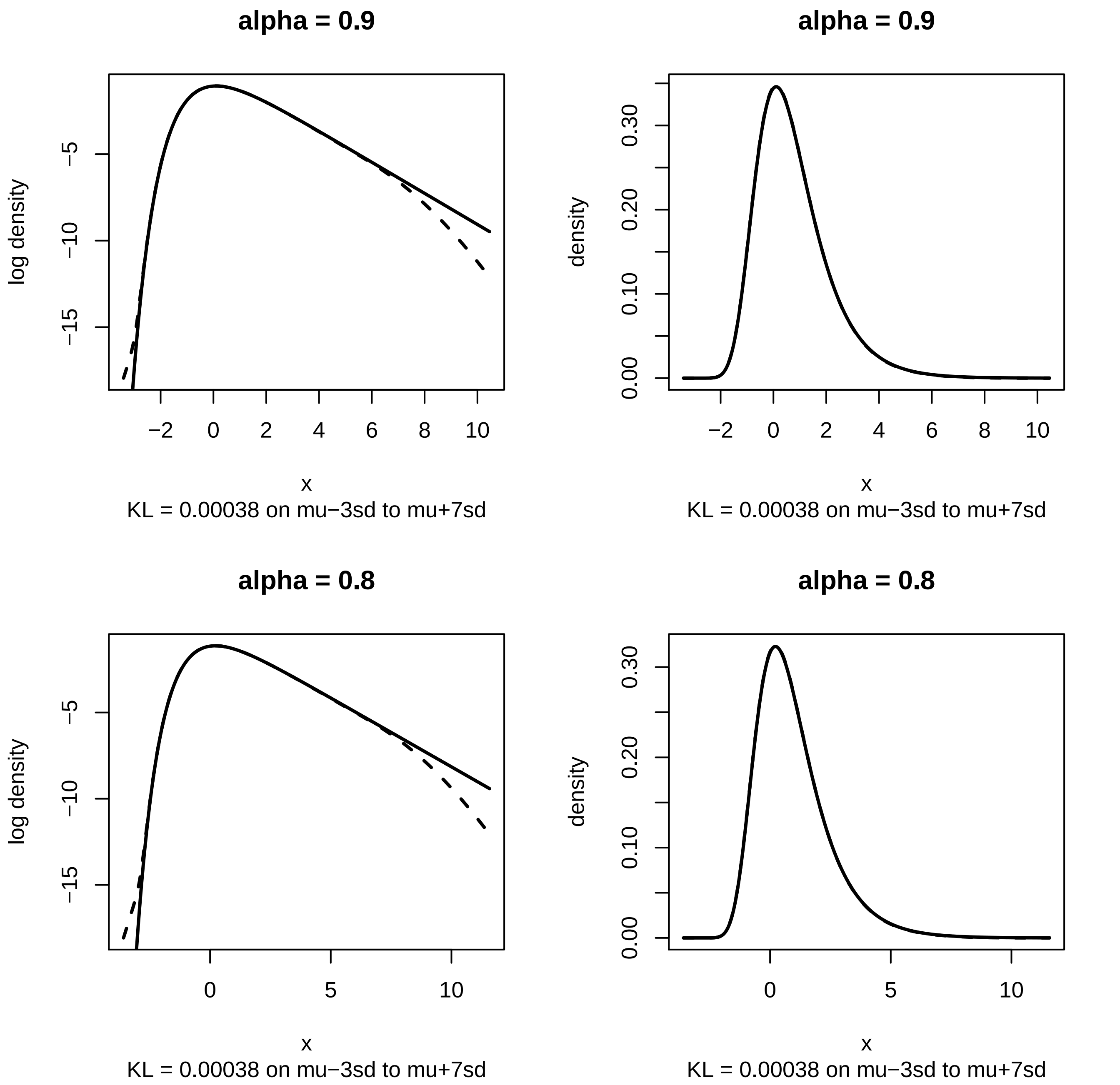
[1] Hugh A. Chipman, Edward I. George, and Robert E. McCulloch. BART: Bayesian Additive Regression Trees. *The Annals of Applied Statistics*, 4(1):266–298, 2010.

[2] S. Fruhwirth-Schnatter, R. Fruhwirth, L. Held, and H. Rue. Improved auxiliary mixture sampling for hierarchical models of non-Gaussian data. *Statistics and computing*, 19:479–492, 2009.

[3] S Kullback and RA Leibler. On information and sufficiency. *Ann Math Stat*, 1951.

[4] J. A. Nelder and R. Mead. A simplex method for function minimization. *The Computer Journal*, 7:308–13, 1965.

[5] T Rowan. *Functional Stability Analysis of Numerical Algorithms*. PhD thesis, Department of Computer Sciences, University of Texas at Austin, 1990.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4
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8
10
x
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0.10
0.20
0.30
−2
0
2
4
6
8
10
x
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log density
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−15
0
5
10
x
KL = 0.00038 on mu−3sd to mu+7sd
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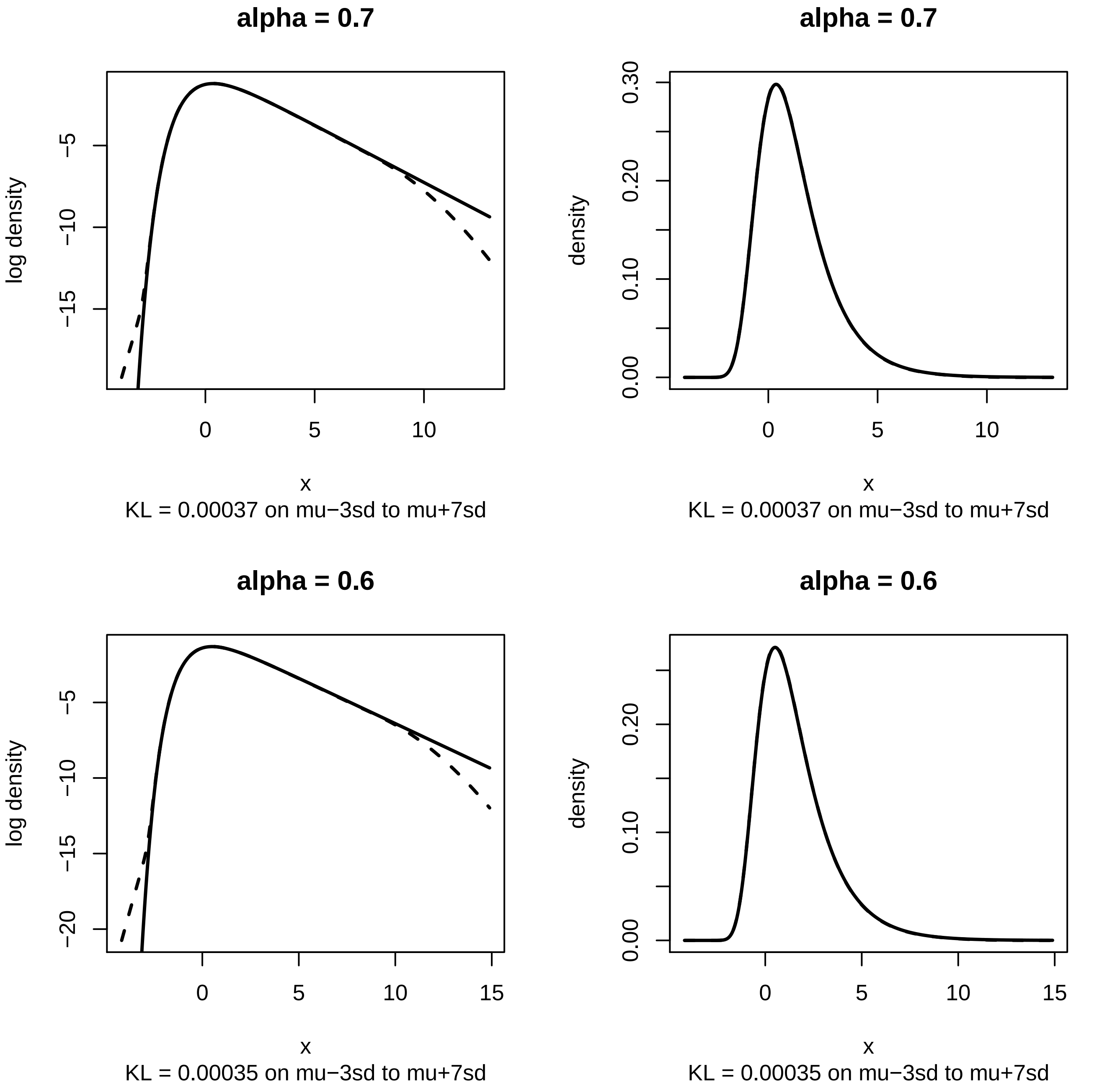
0
5
10
x
KL = 0.00038 on mu−3sd to mu+7sd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density
0.30
0.20
0.10
0.00
0
5
10
x
KL = 0.00037 on mu−3sd to mu+7sd
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log density
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−10
−15
−20
0
5
10
15
x
KL = 0.00035 on mu−3sd to mu+7sd
alpha = 0.6
density
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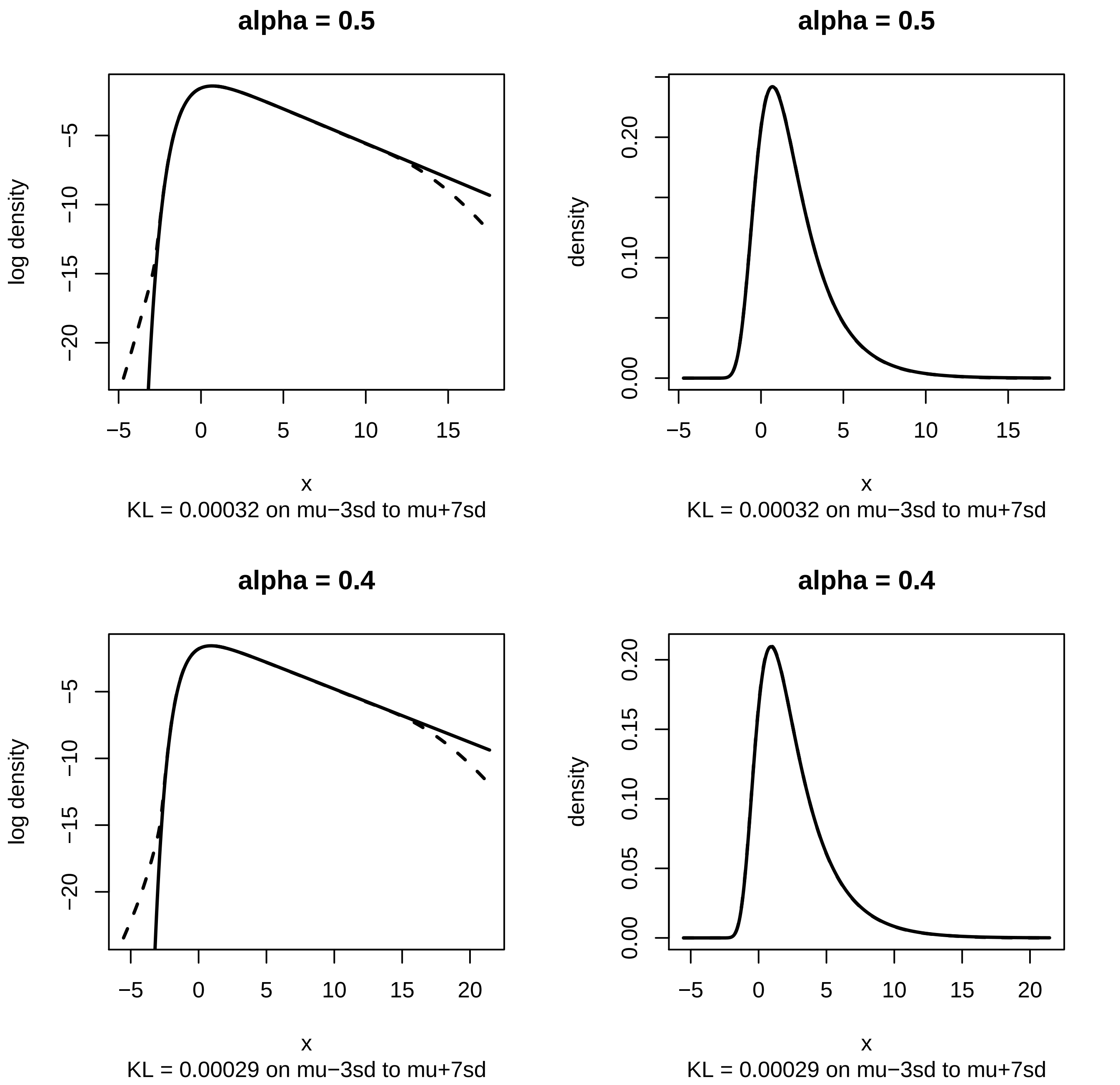
0.10
0.00
0
5
10
15
x
KL = 0.00035 on mu−3sd to mu+7sd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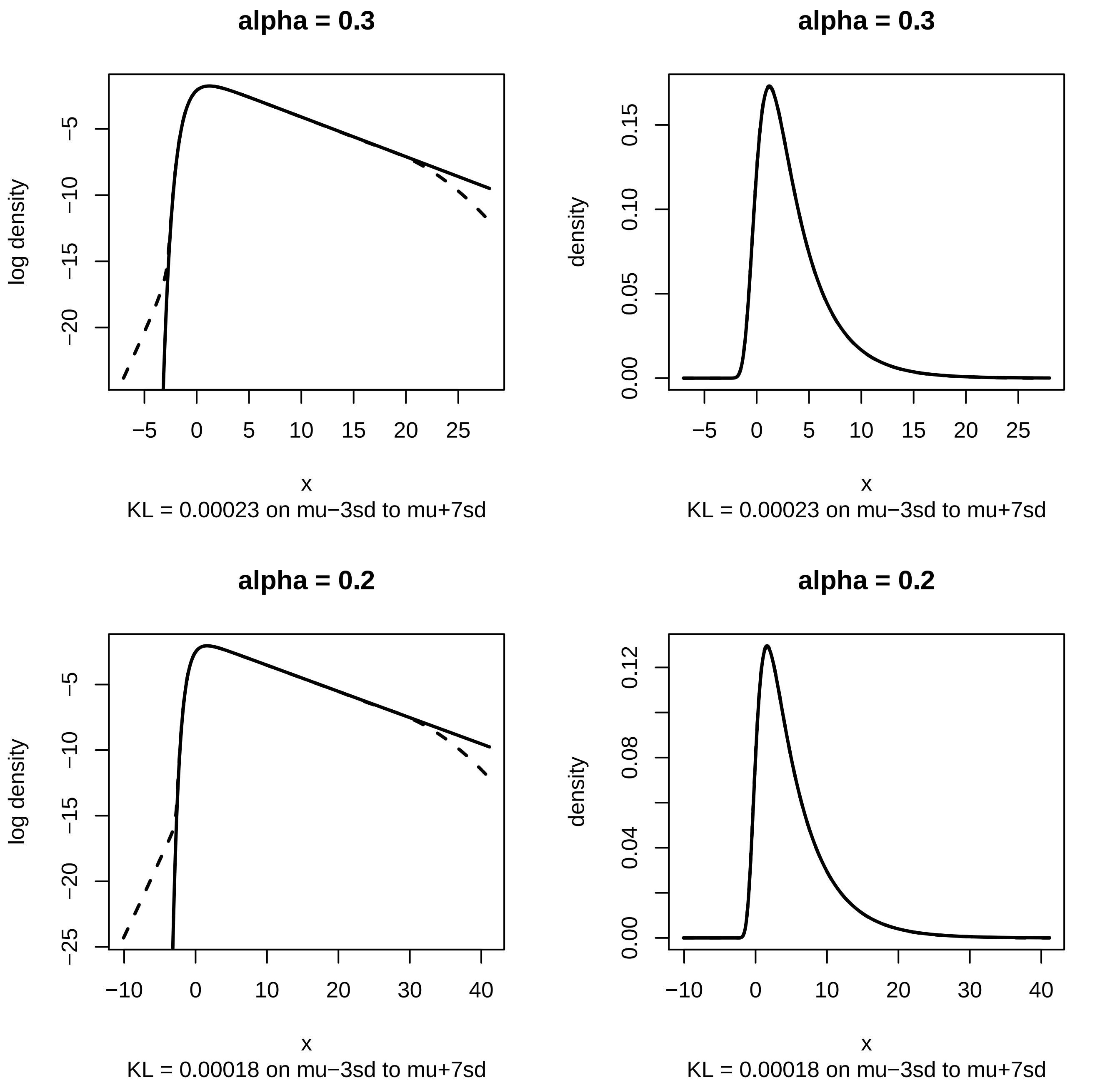
KL = 0.00032 on mu−3sd to mu+7sd
alpha = 0.4
log density
x
KL = 0.00029 on mu−3sd to mu+7sd
alpha = 0.4
density
x
KL = 0.00029 on mu−3sd to mu+7sd



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−10
−15
−20
−5 0 5 10 15 20 25
x
KL = 0.00023 on mu−3sd to mu+7sd
alpha = 0.3
density
0.00 0.05 0.10 0.15
−5 0 5 10 15 20 25
x
KL = 0.00023 on mu−3sd to mu+7sd
alpha = 0.2
log density
−5
−10
−15
−20
−25
−10 0 10 20 30 40
x
KL = 0.00018 on mu−3sd to mu+7sd
alpha = 0.2
density
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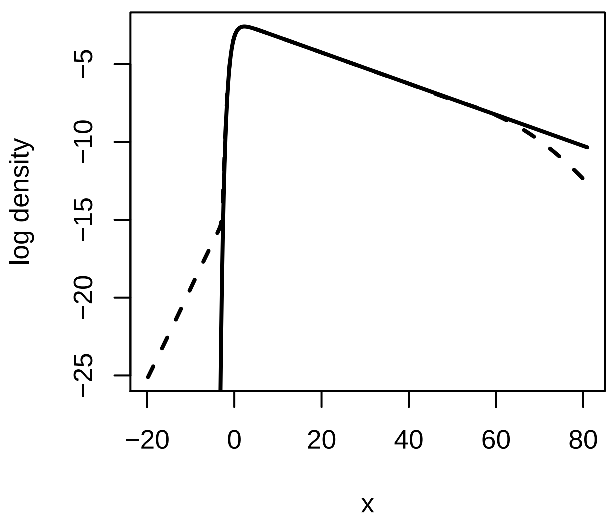
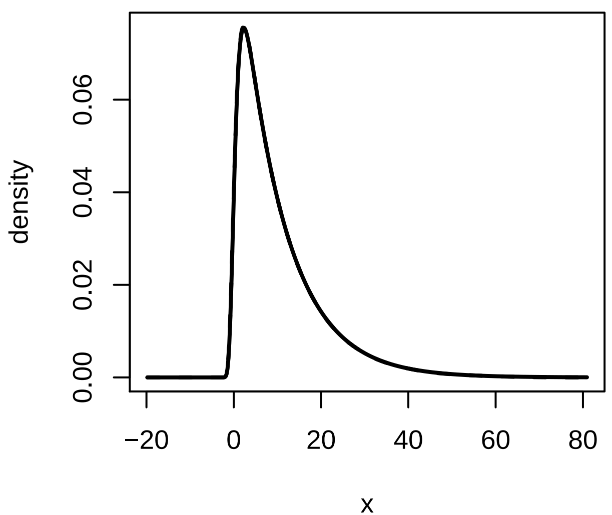
−10 0 10 20 30 40
x
KL = 0.00018 on mu−3sd to mu+7sd

alpha = 0.1



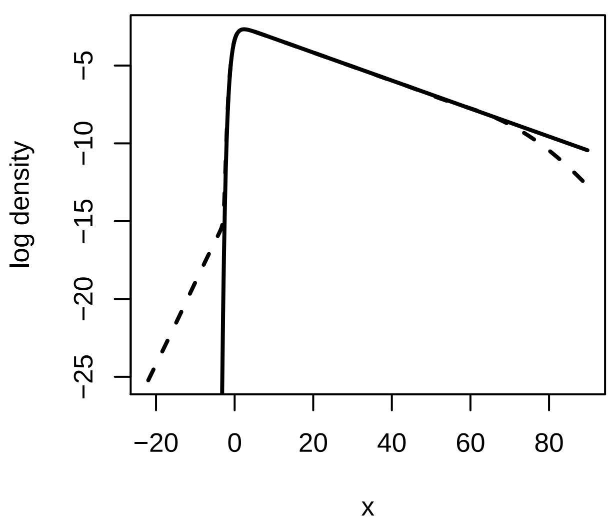
KL = 0.00015 on mu−3sd to mu+7sd

alpha = 0.1



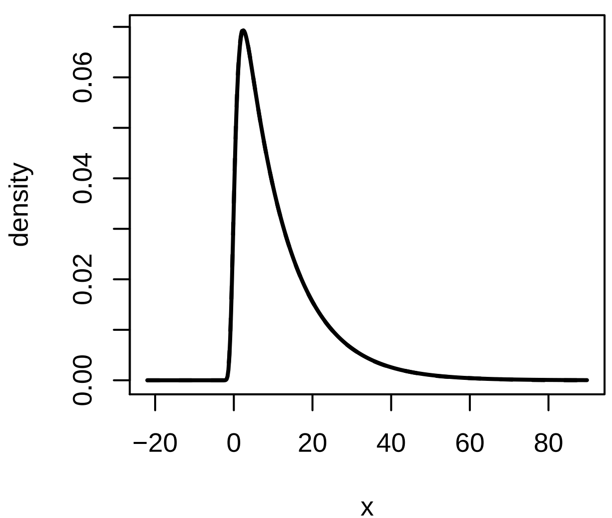
KL = 0.00015 on mu−3sd to mu+7sd

alpha = 0.09

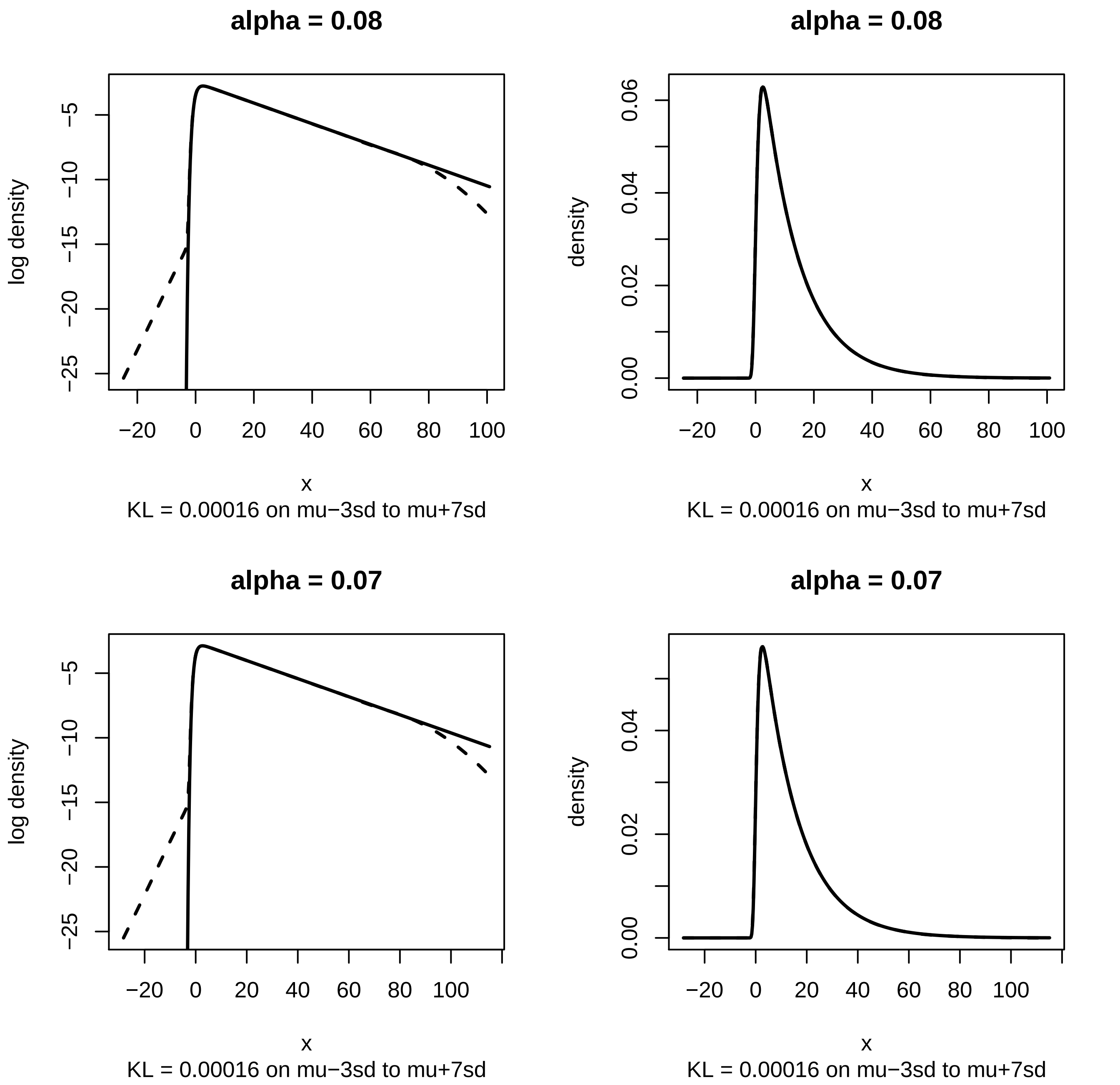


KL = 0.00015 on mu−3sd to mu+7sd

alpha = 0.09



KL = 0.00015 on mu−3sd to mu+7sd


alpha = 0.08
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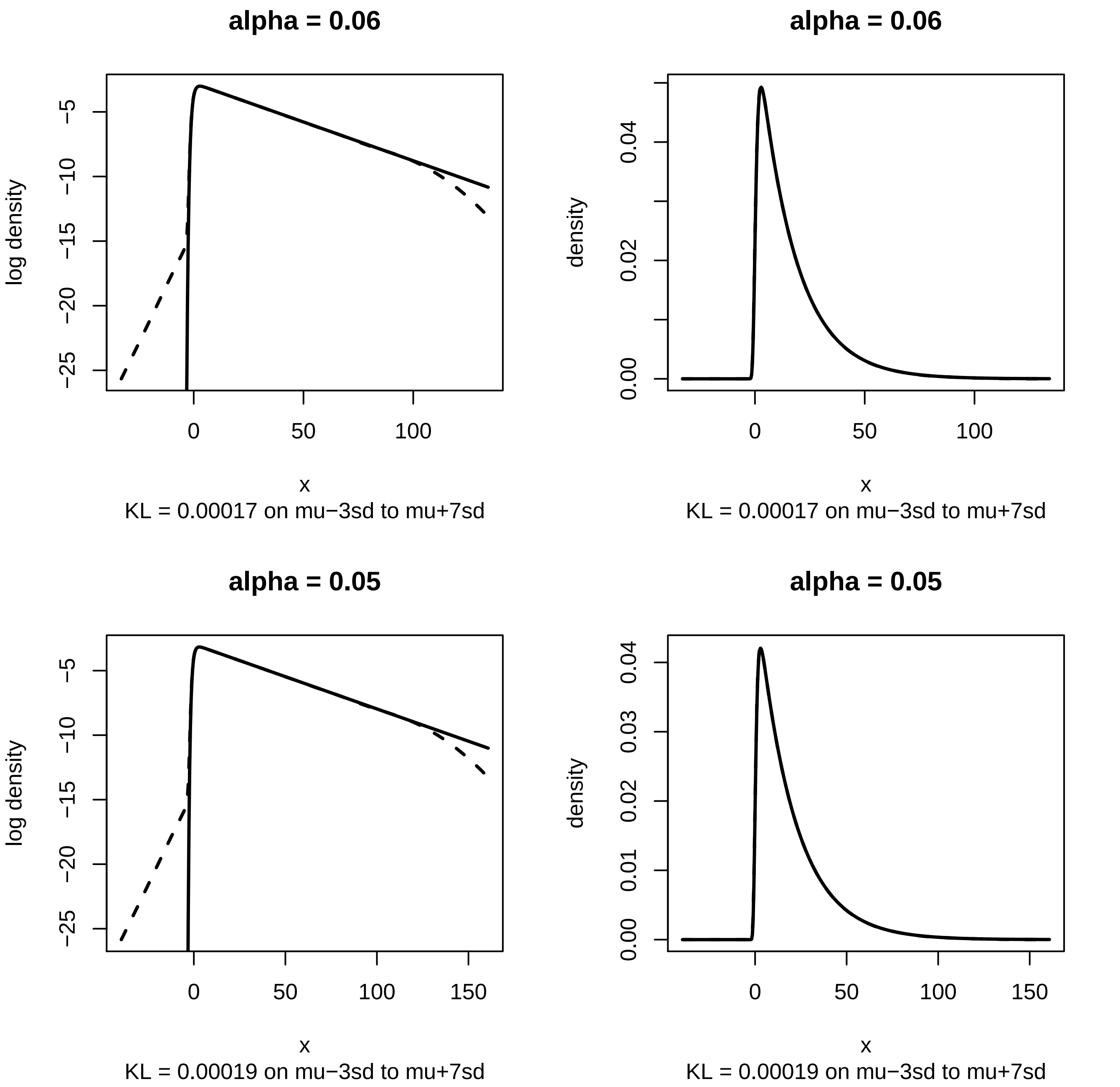
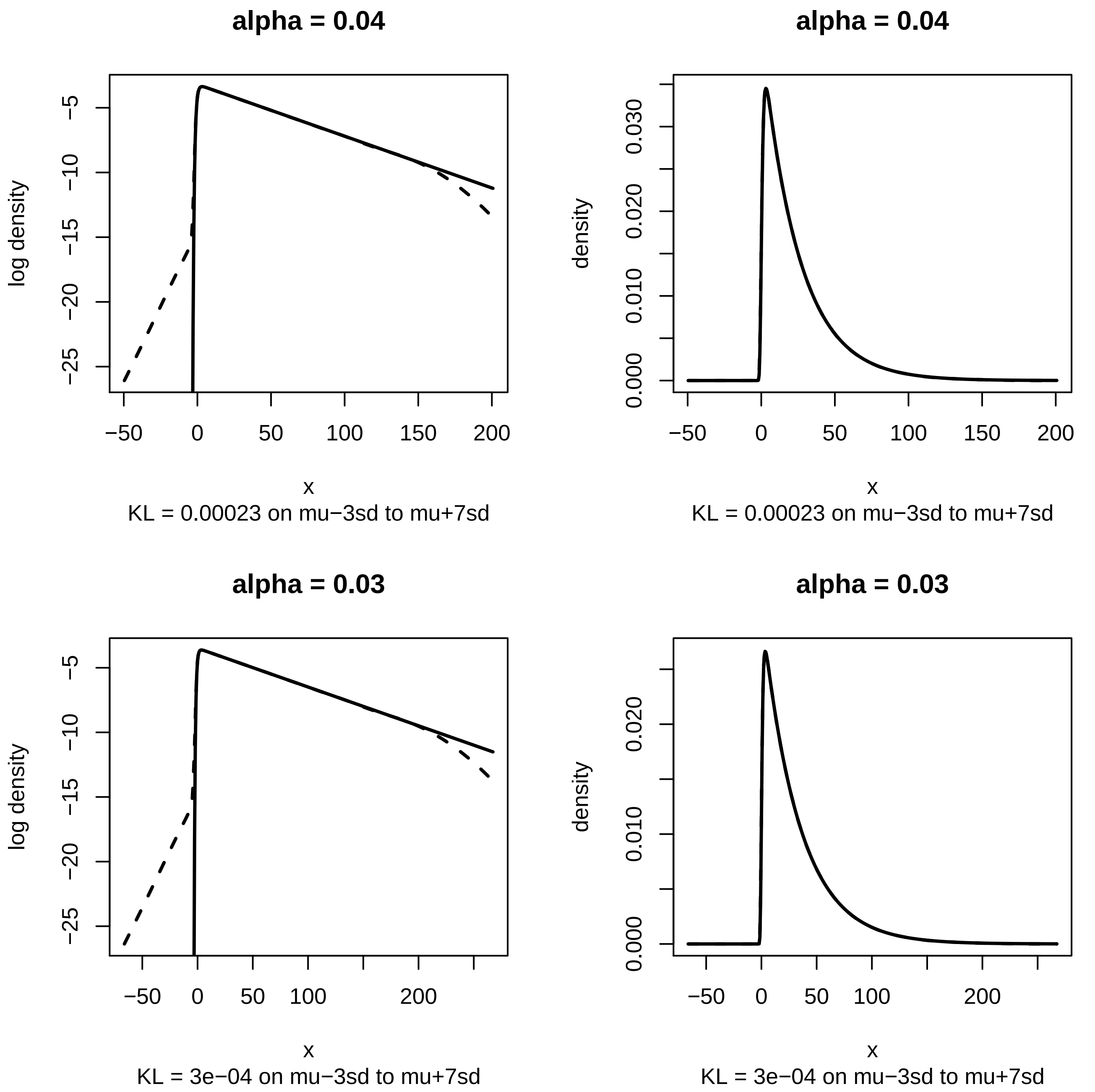
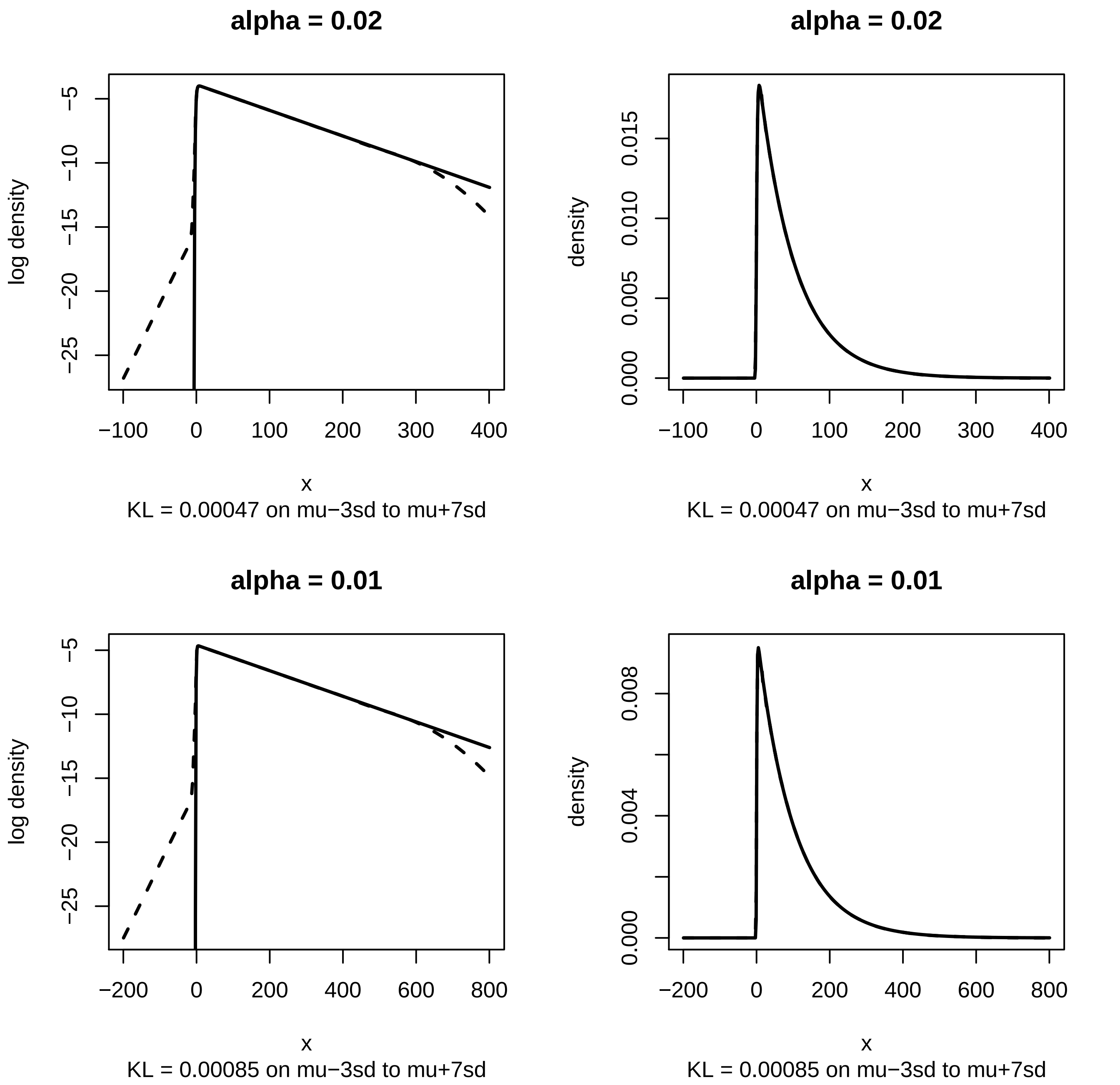
-10
-15
-20
-25
-20
0
20
40
60
80
100
x
KL = 0.00016 on mu−3sd to mu+7sd
alpha = 0.08
density
0.06
0.04
0.02
0.00
-20
0
20
40
60
80
100
x
KL = 0.00016 on mu−3sd to mu+7sd
alpha = 0.07
log density
-5
-10
-15
-20
-25
-20
0
20
40
60
80
100
x
KL = 0.00016 on mu−3sd to mu+7sd
alpha = 0.07
density
0.04
0.02
0.00
-20
0
20
40
60
80
100
x
KL = 0.00016 on mu−3sd to mu+7sd


alpha = 0.06
log density
x
KL = 0.00017 on mu−3sd to mu+7sd
alpha = 0.06
density
x
KL = 0.00017 on mu−3sd to mu+7sd
alpha = 0.05
log density
x
KL = 0.00019 on mu−3sd to mu+7sd
alpha = 0.05
density
x
KL = 0.00019 on mu−3sd to mu+7sd


alpha = 0.04
log density
x
KL = 0.00023 on mu−3sd to mu+7sd
alpha = 0.04
density
x
KL = 0.00023 on mu−3sd to mu+7sd
alpha = 0.03
log density
x
KL = 3e−04 on mu−3sd to mu+7sd
alpha = 0.03
density
x
KL = 3e−04 on mu−3sd to mu+7sd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−20
−25
−100
0
100
200
300
400
x
KL = 0.00047 on mu−3sd to mu+7sd
alpha = 0.02
density
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0.005
0.010
0.015
−100
0
100
200
300
400
x
KL = 0.00047 on mu−3sd to mu+7sd
alpha = 0.01
log density
−5
−10
−15
−20
−25
−200
0
200
400
600
800
x
KL = 0.00085 on mu−3sd to mu+7sd
alpha = 0.01
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0.004
0.008
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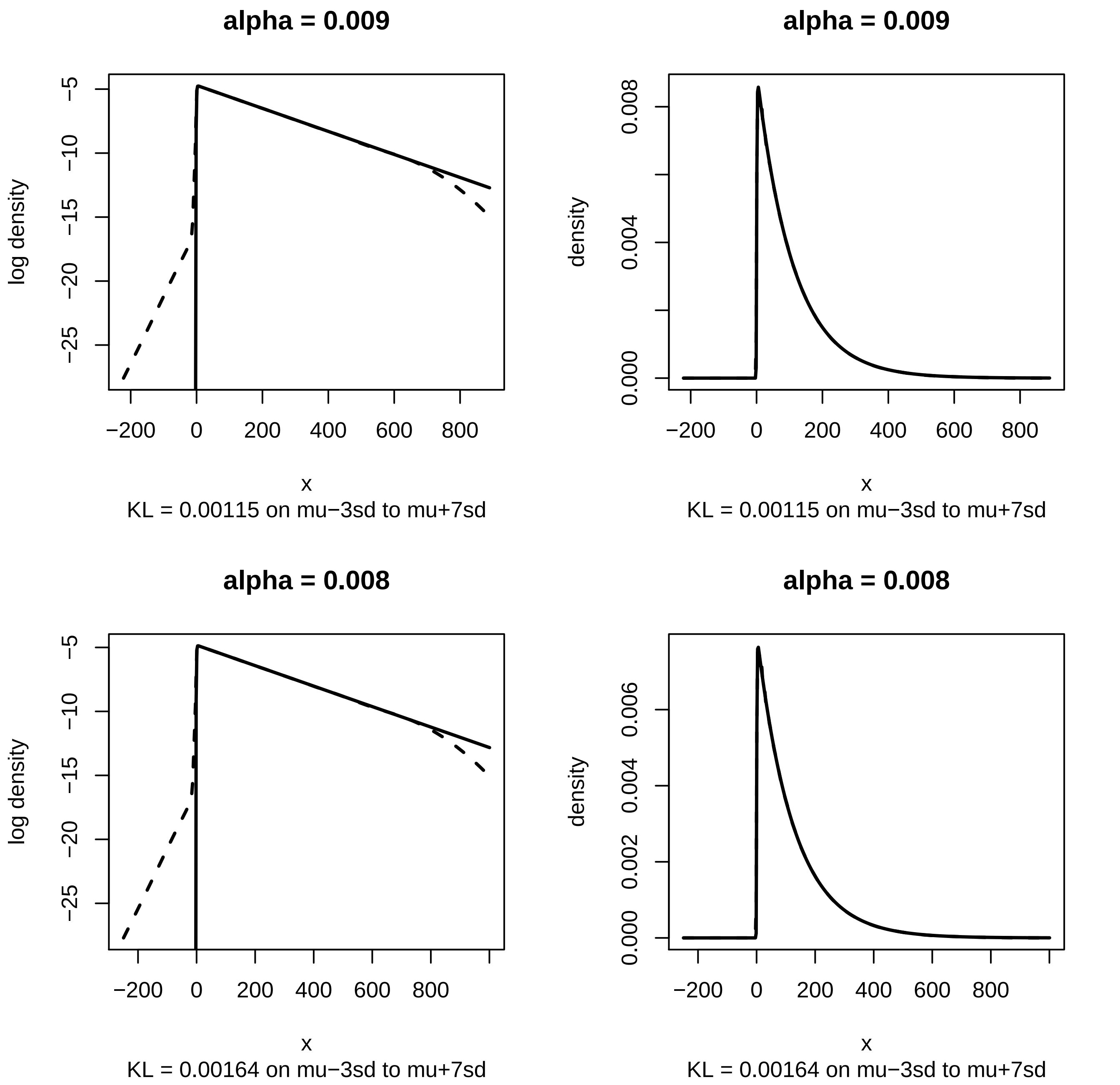
0
200
400
600
800
x
KL = 0.00085 on mu−3sd to mu+7sd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−10
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−20
−25
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0
200
400
600
800
x
KL = 0.00115 on mu−3sd to mu+7sd
alpha = 0.009
density
0.000
0.004
0.008
−200
0
200
400
600
800
x
KL = 0.00115 on mu−3sd to mu+7sd
alpha = 0.008
log density
−5
−10
−15
−20
−25
−200
0
200
400
600
800
x
KL = 0.00164 on mu−3sd to mu+7sd
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400
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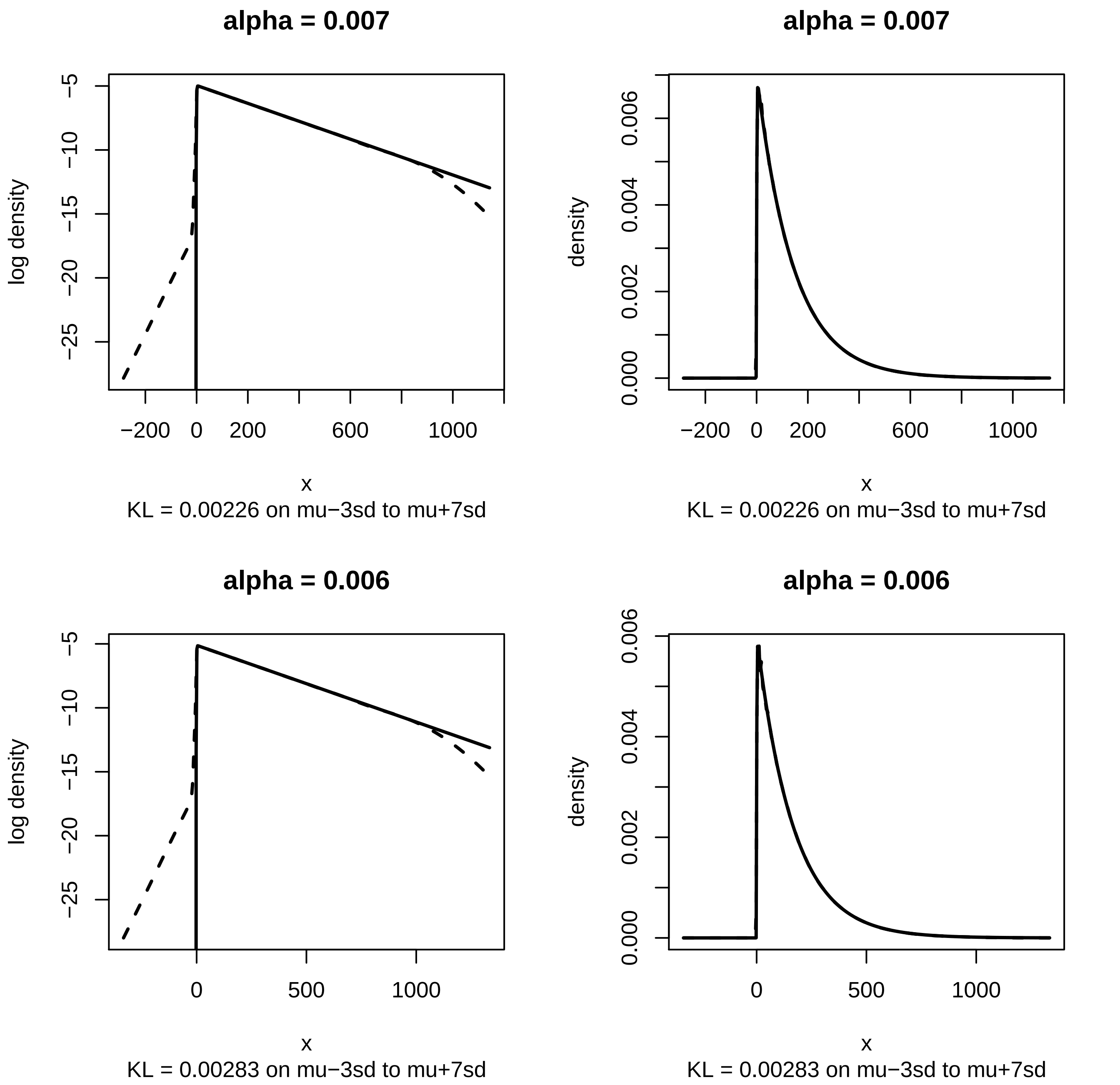
800
x
KL = 0.00164 on mu−3sd to mu+7sd



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−20
−25
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0
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600
1000
x
KL = 0.00226 on mu−3sd to mu+7sd
alpha = 0.007
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0.002
0.004
0.006
−200
0
200
600
1000
x
KL = 0.00226 on mu−3sd to mu+7sd
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log density
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500
1000
x
KL = 0.00283 on mu−3sd to mu+7sd
alpha = 0.006
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x
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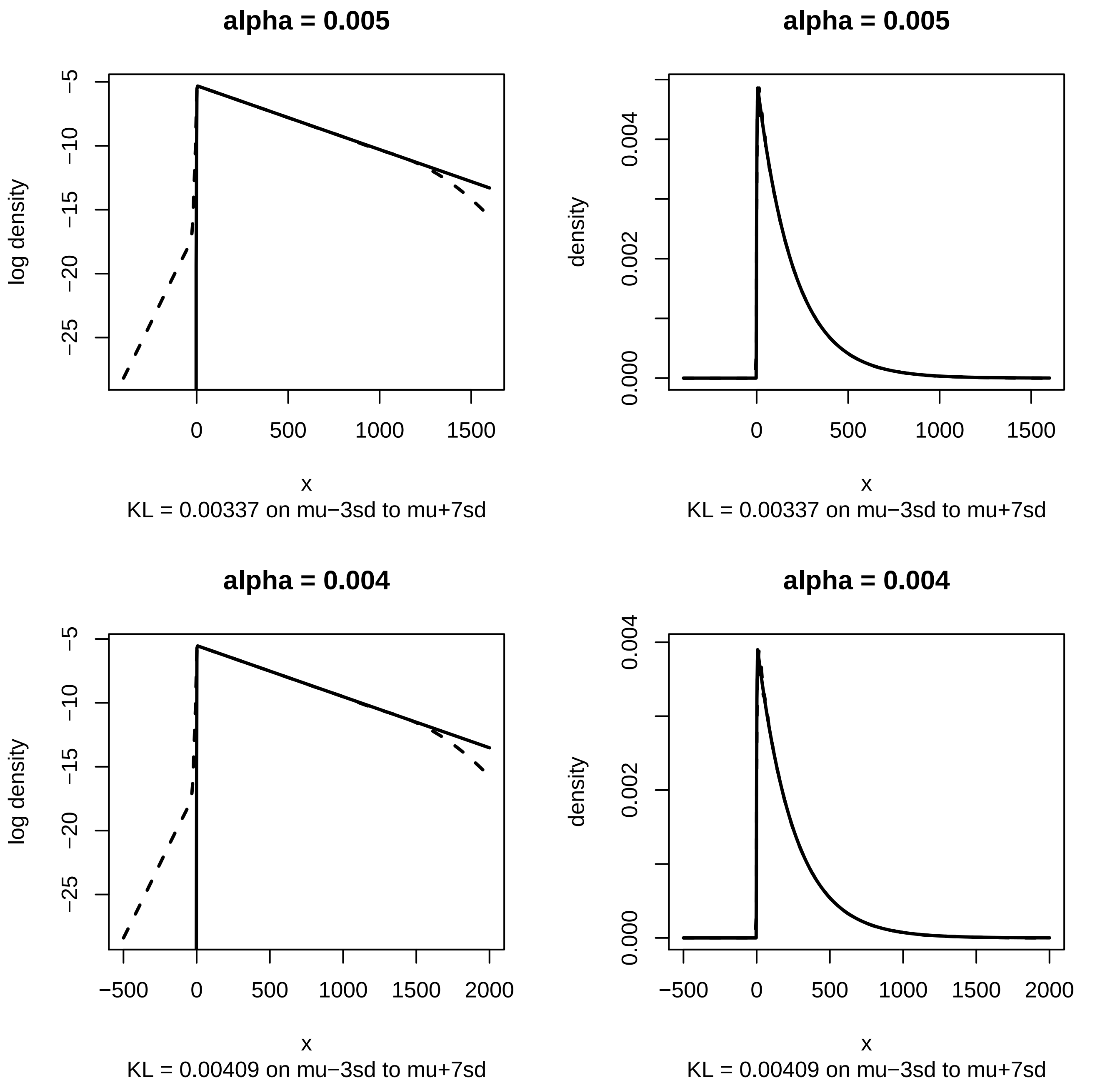

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density
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1500
x
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alpha = 0.004
log density
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1000
1500
2000
x
KL = 0.00409 on mu−3sd to mu+7sd
alpha = 0.004
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2000
x
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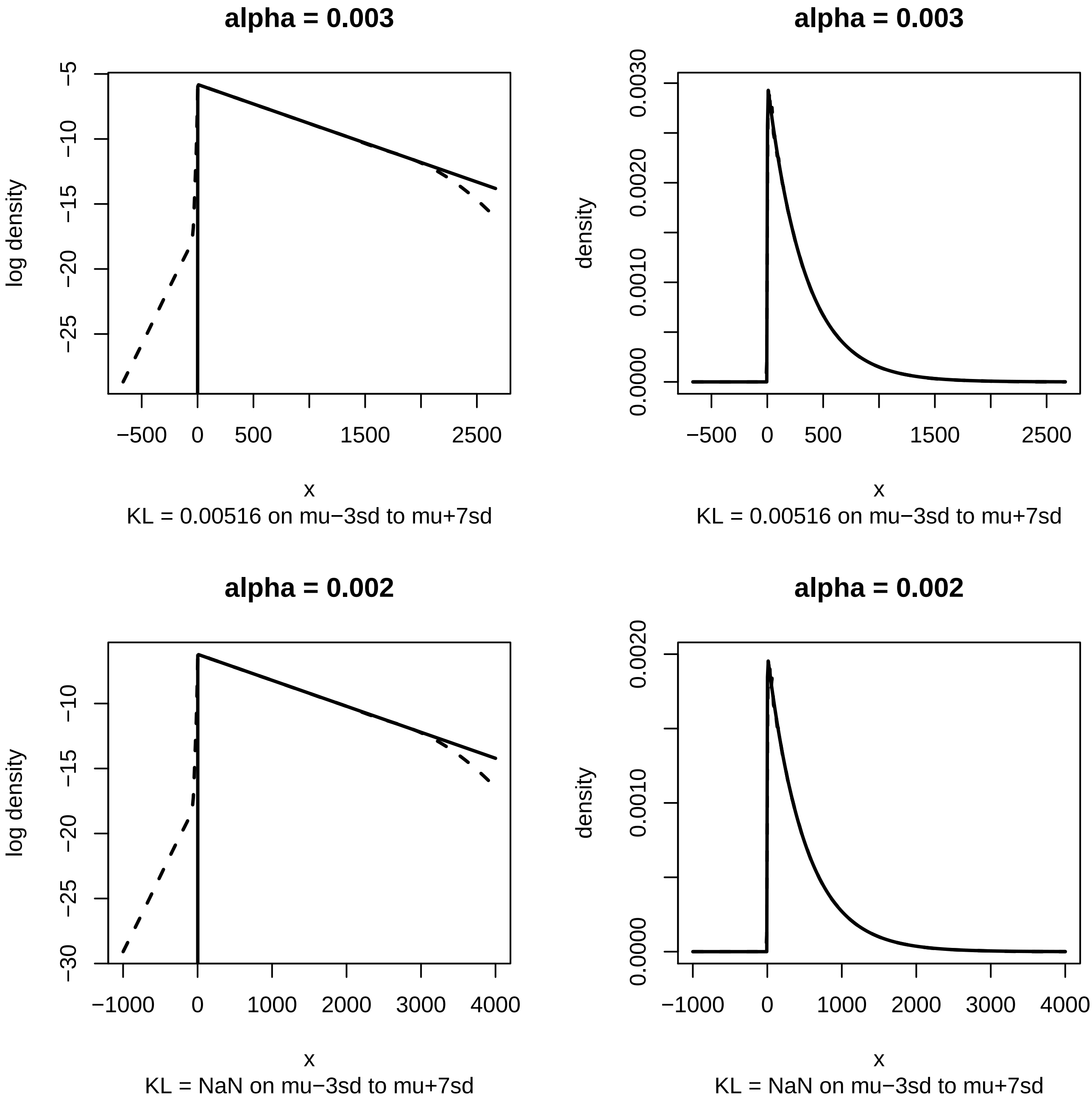
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500
1500
2500
x
KL = 0.00516 on mu−3sd to mu+7sd
alpha = 0.003
density
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0.0010
0.0020
0.0030
−500
0
500
1500
2500
x
KL = 0.00516 on mu−3sd to mu+7sd
alpha = 0.002
log density
−10
−15
−20
−25
−30
−1000
0
1000
2000
3000
4000
x
KL = NaN on mu−3sd to mu+7sd
alpha = 0.002
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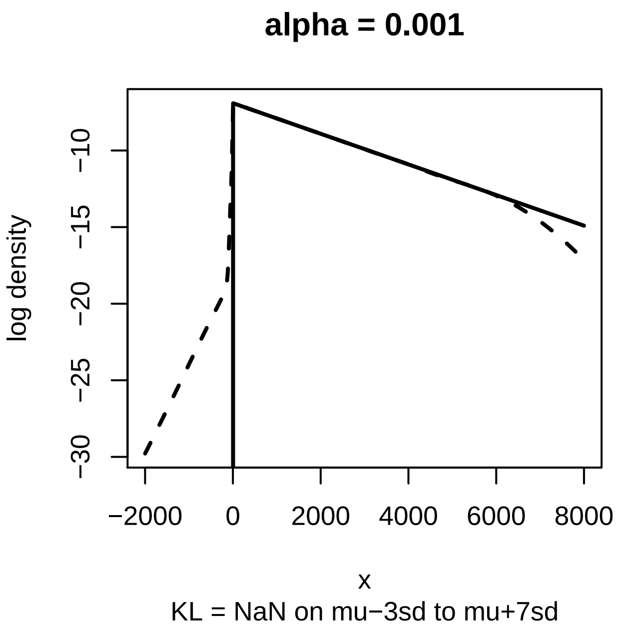
2000
3000
4000
x
KL = NaN on mu−3sd to mu+7sd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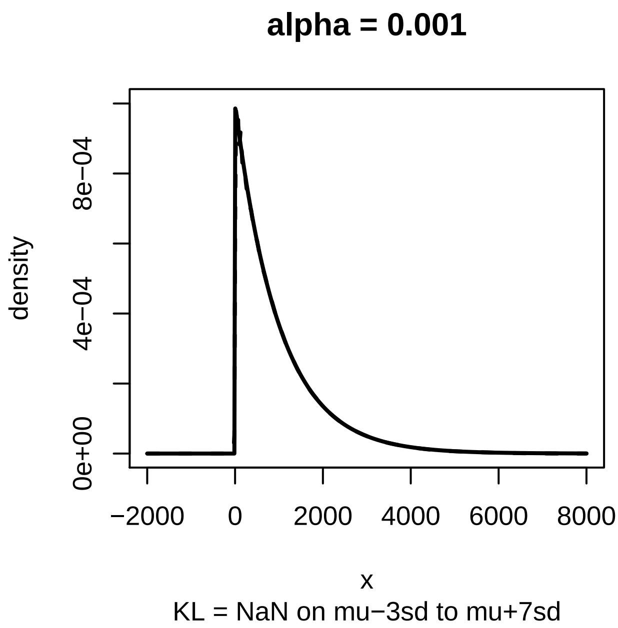
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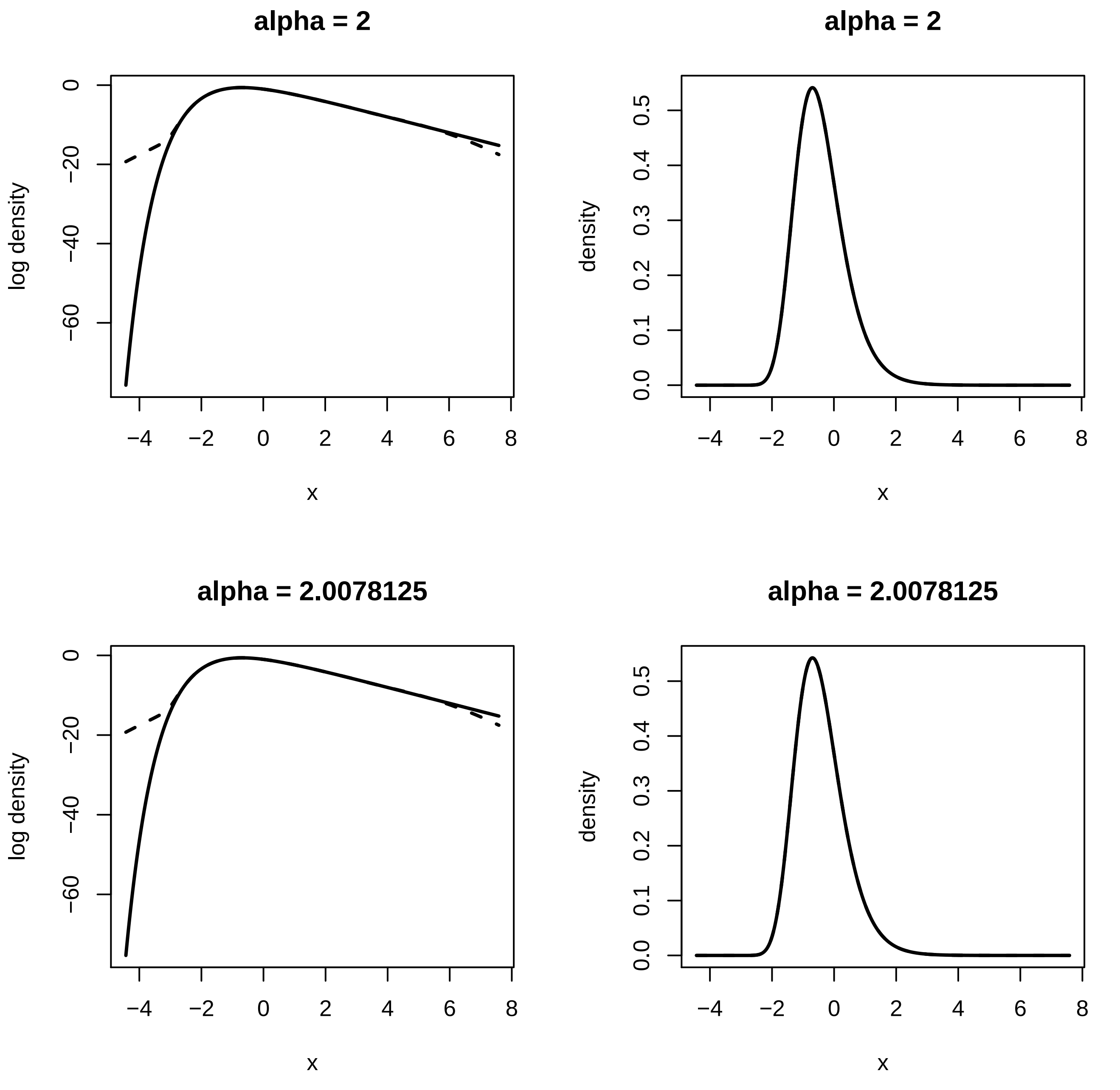
KL = NaN on mu−3sd to mu+7sd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x
alpha = 2
density
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log density
x
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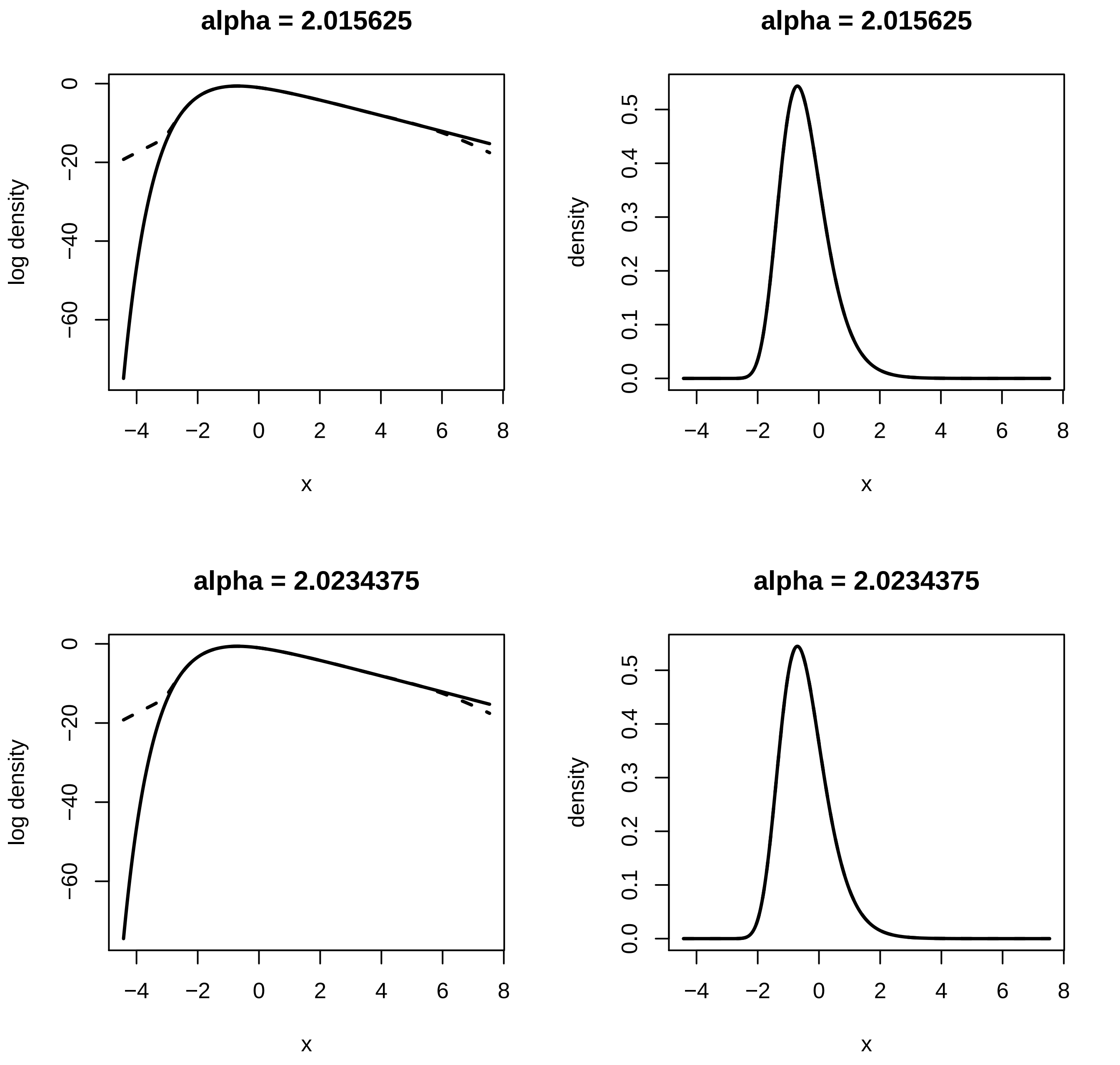
density
x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0
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4
6
8
x
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density
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0.1
0.2
0.3
0.4
0.5
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−2
0
2
4
6
8
x
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log density
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2
4
6
8
x
alpha = 2.0234375
density
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−2
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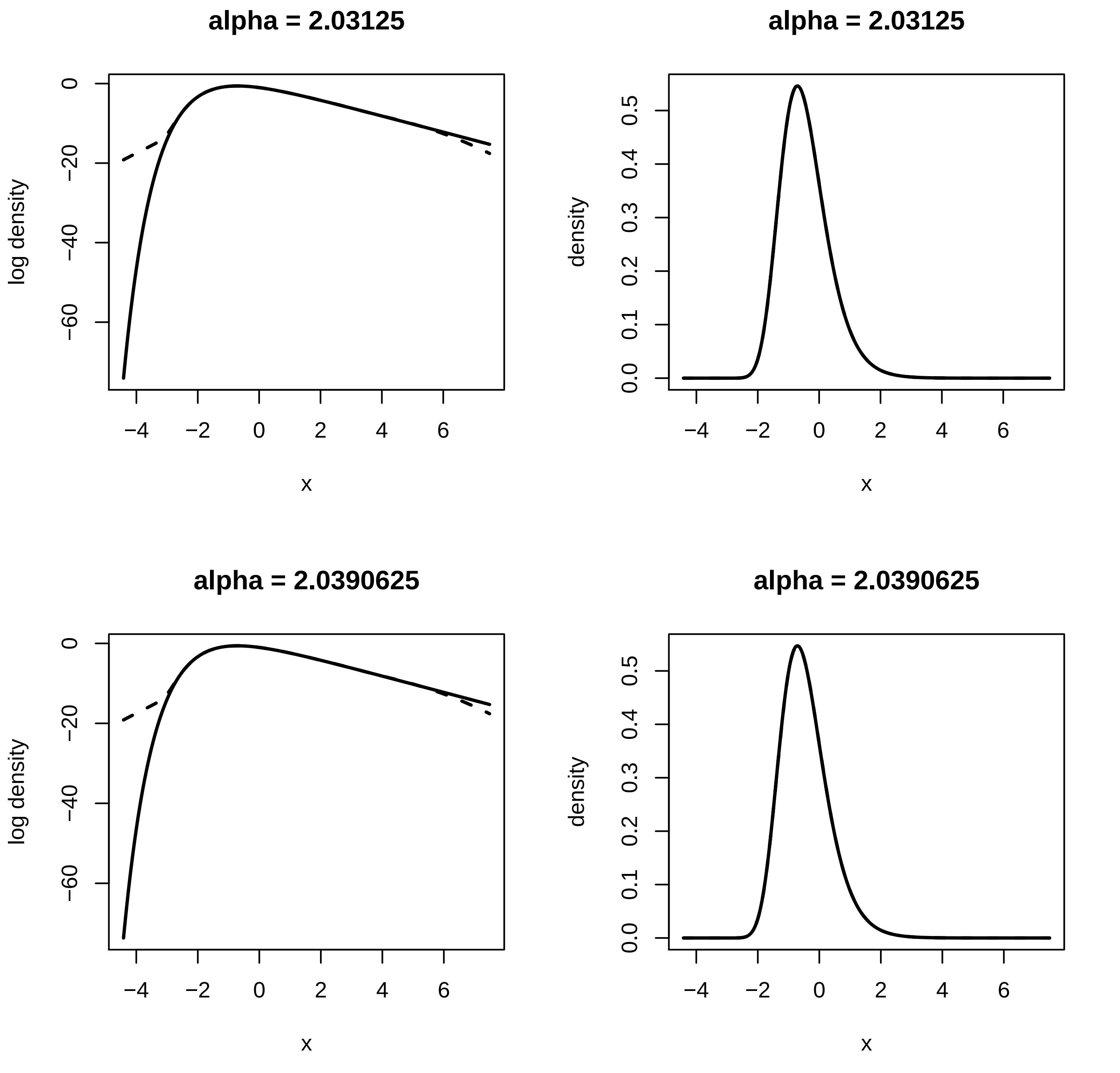
2
4
6
8
x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2
4
6
x
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density
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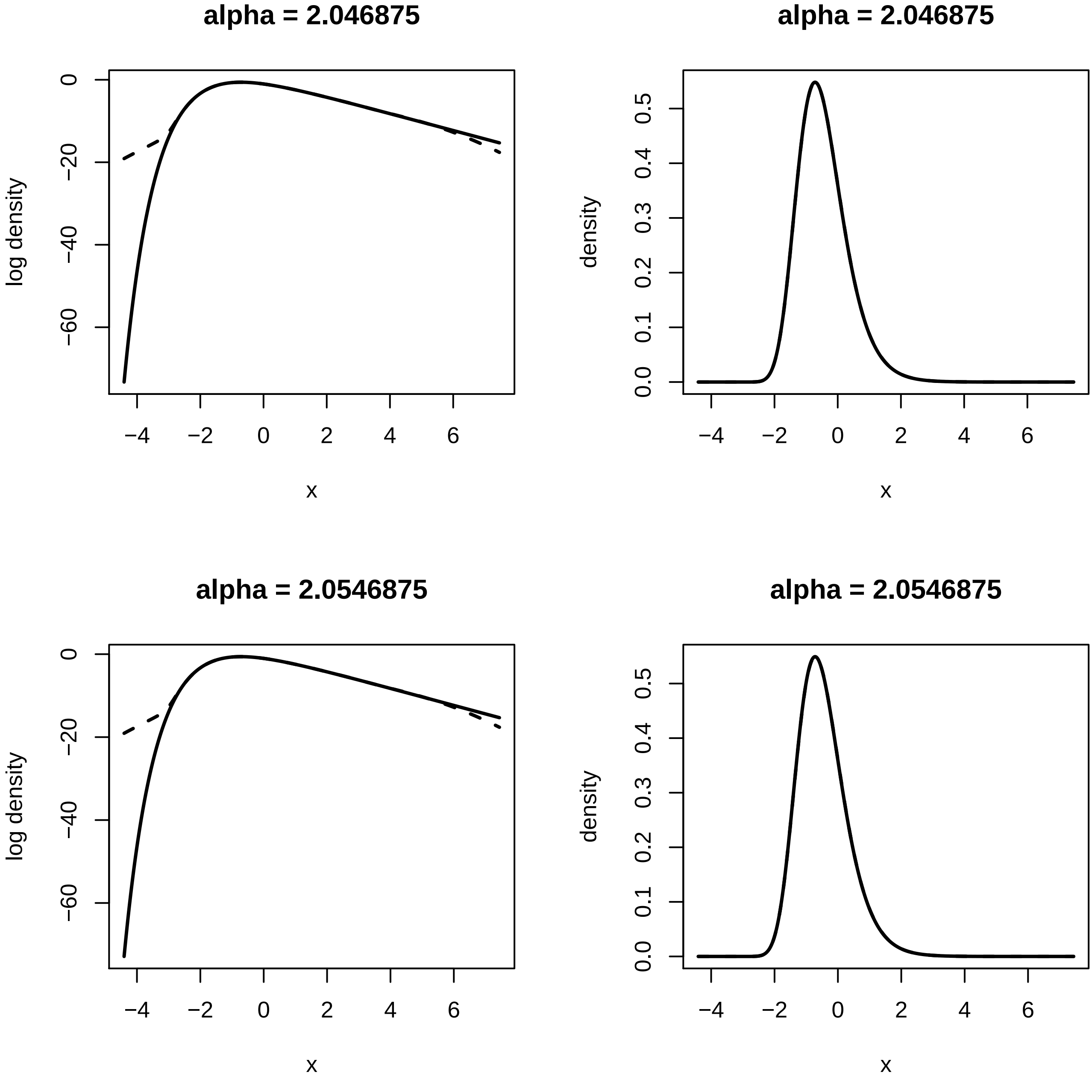
0.2
0.3
0.4
0.5
x
alpha = 2.0390625
log density
x
alpha = 2.0390625
density
x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4
6
x
alpha = 2.046875
density
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0.1
0.2
0.3
0.4
0.5
−4
−2
0
2
4
6
x
alpha = 2.0546875
log density
0
−20
−40
−60
−4
−2
0
2
4
6
x
alpha = 2.0546875
density
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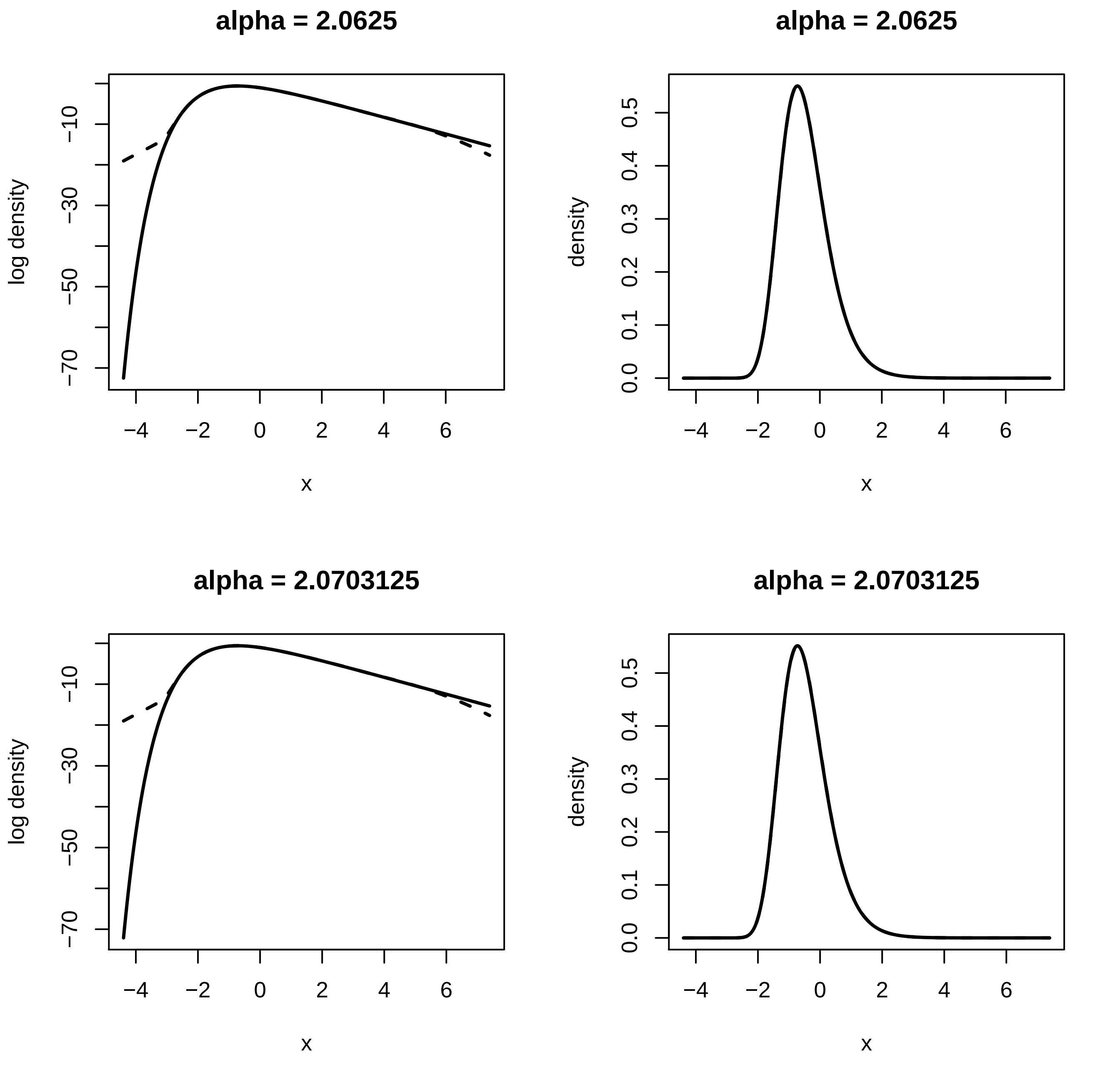
0.3
0.4
0.5
−4
−2
0
2
4
6
x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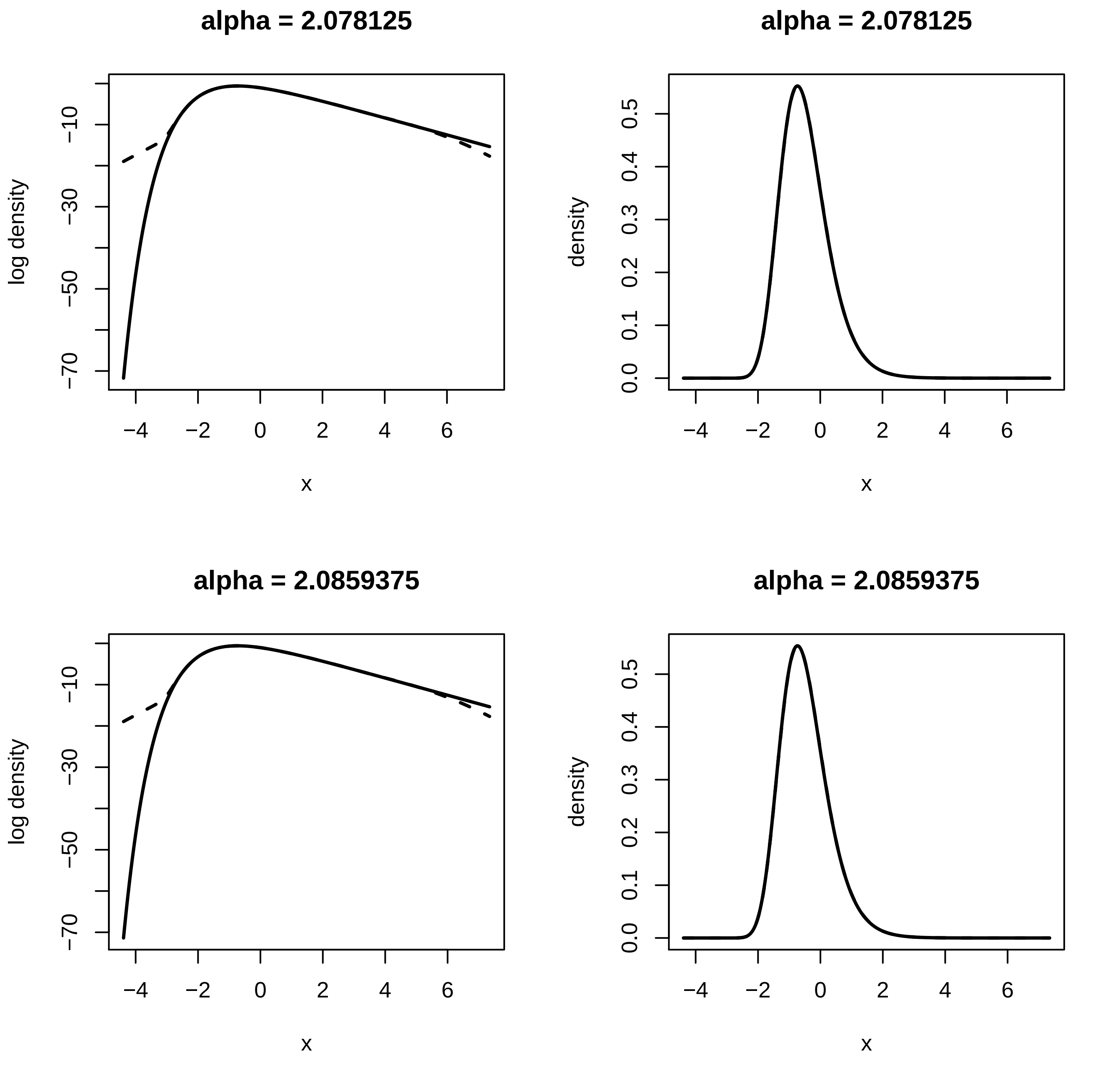
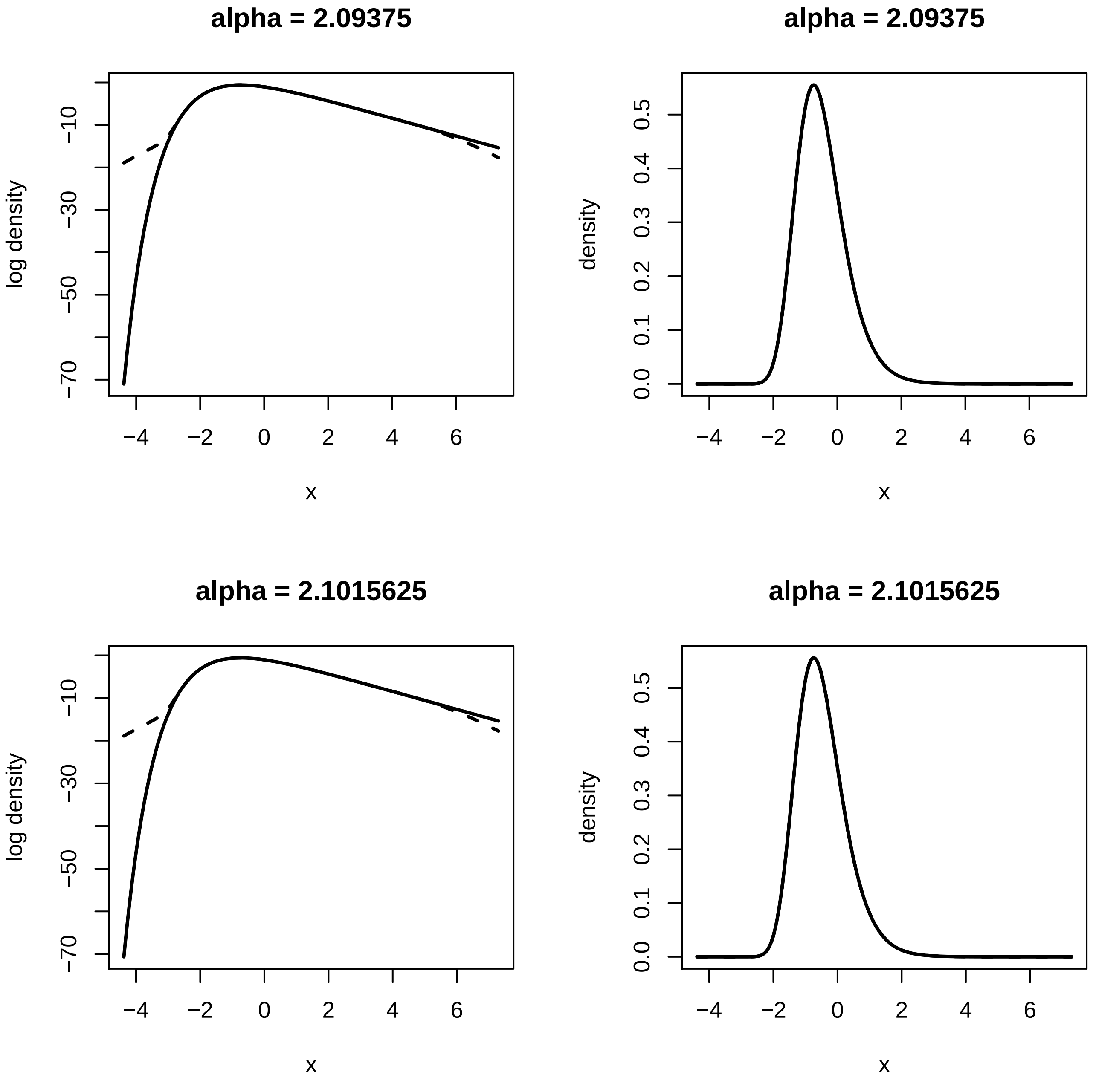
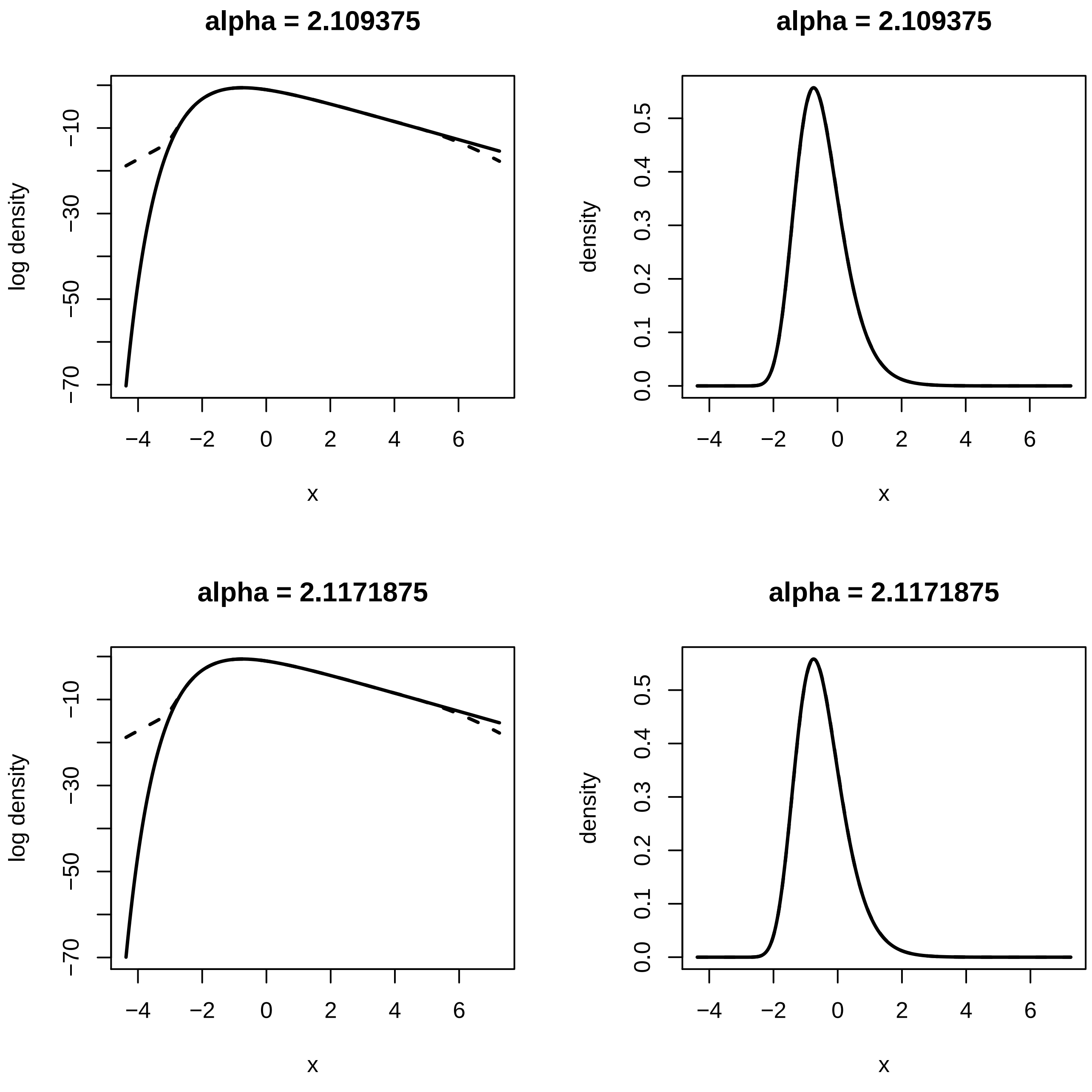
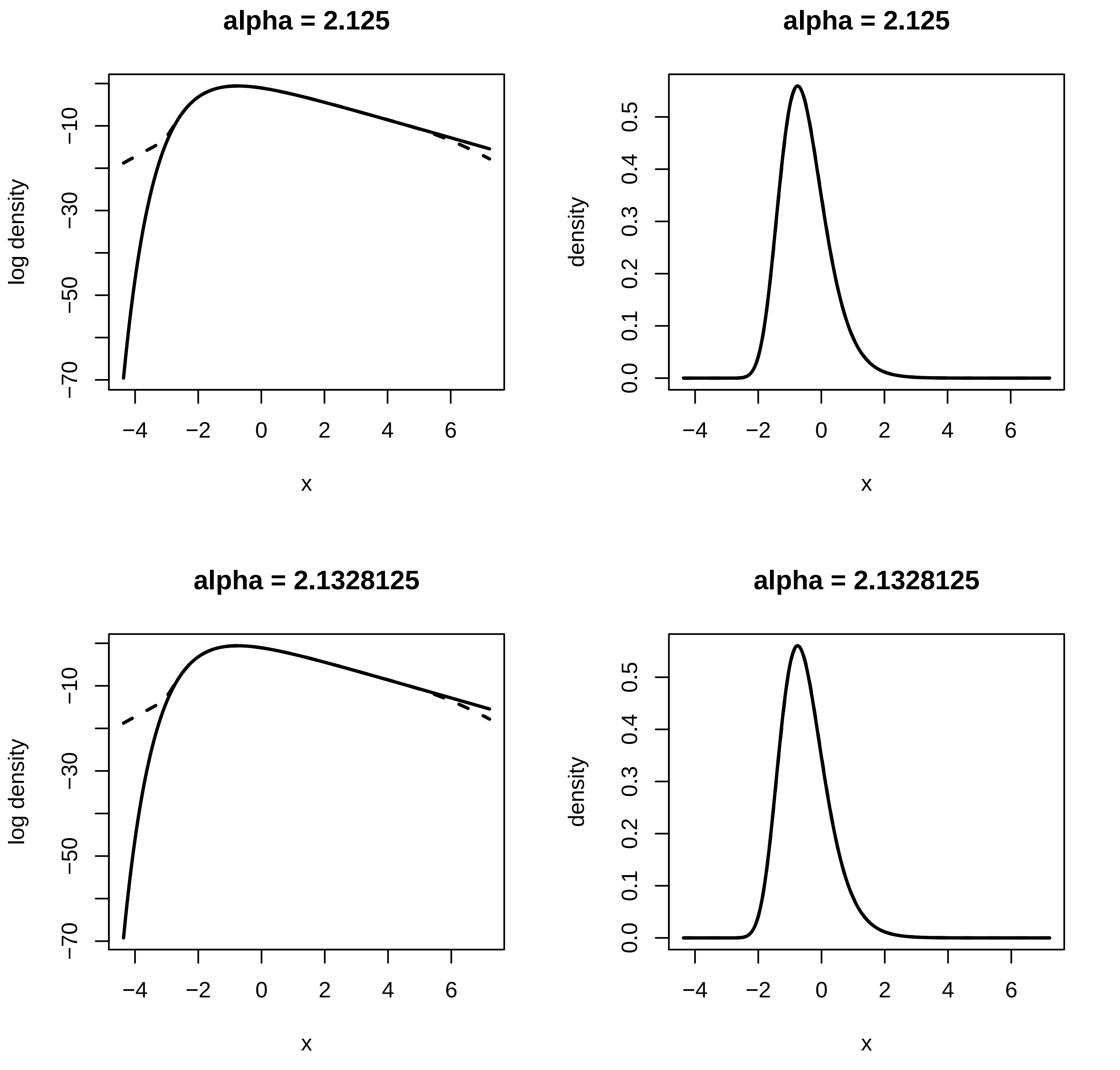
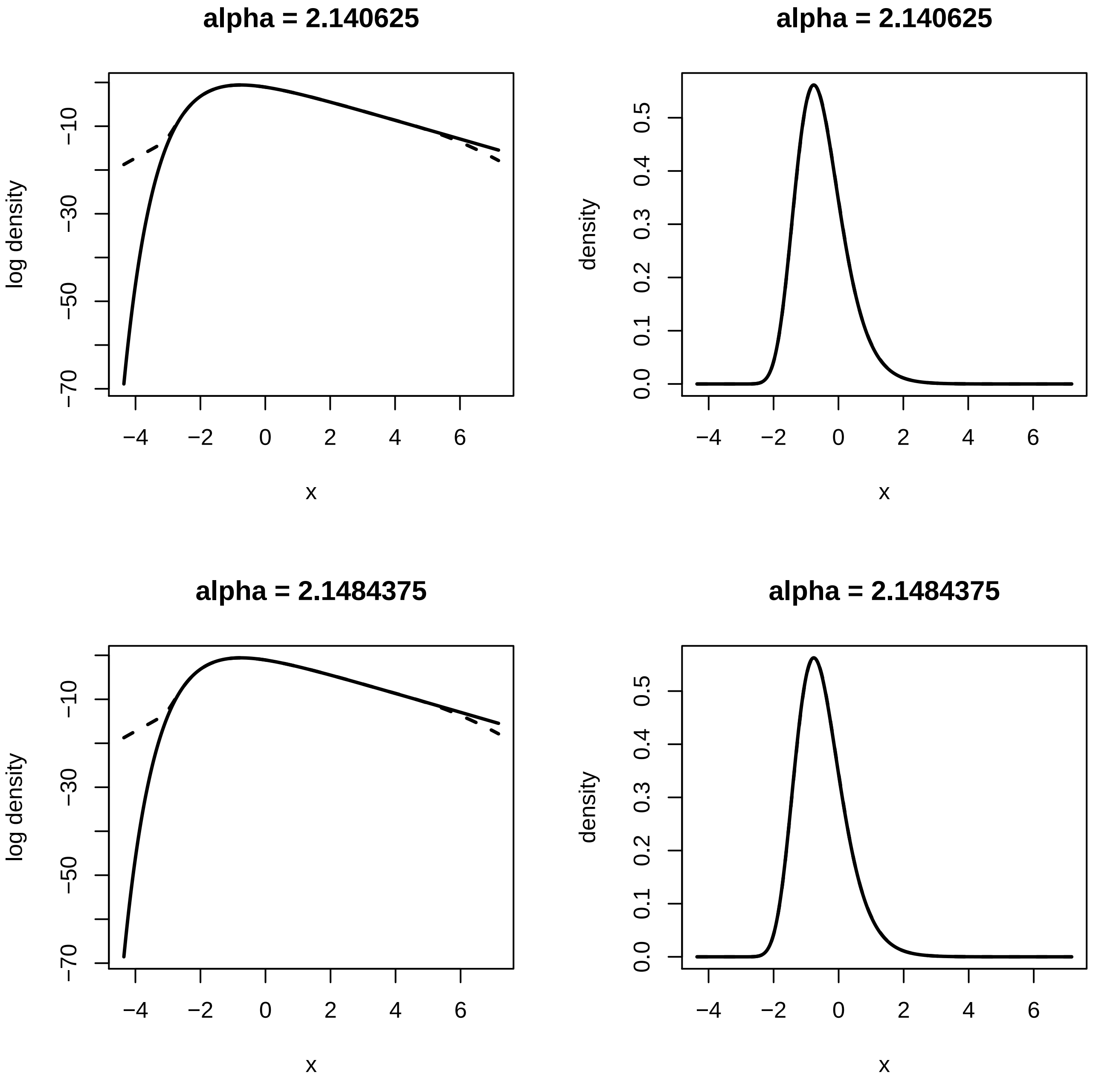
−30
−50
−70
−4
−2
0
2
4
6
x
alpha = 2.0625
density
0.0
0.1
0.2
0.3
0.4
0.5
−4
−2
0
2
4
6
x
alpha = 2.0703125
log density
−10
−30
−50
−70
−4
−2
0
2
4
6
x
alpha = 2.0703125
density
0.0
0.1
0.2
0.3
0.4
0.5
−4
−2
0
2
4
6
x


alpha = 2.078125
alpha = 2.078125
alpha = 2.0859375
alpha = 2.0859375
log density
density
x
−10
−30
−50
−70
0.0
0.1
0.2
0.3
0.4
0.5
−4
−2
0
2
4
6

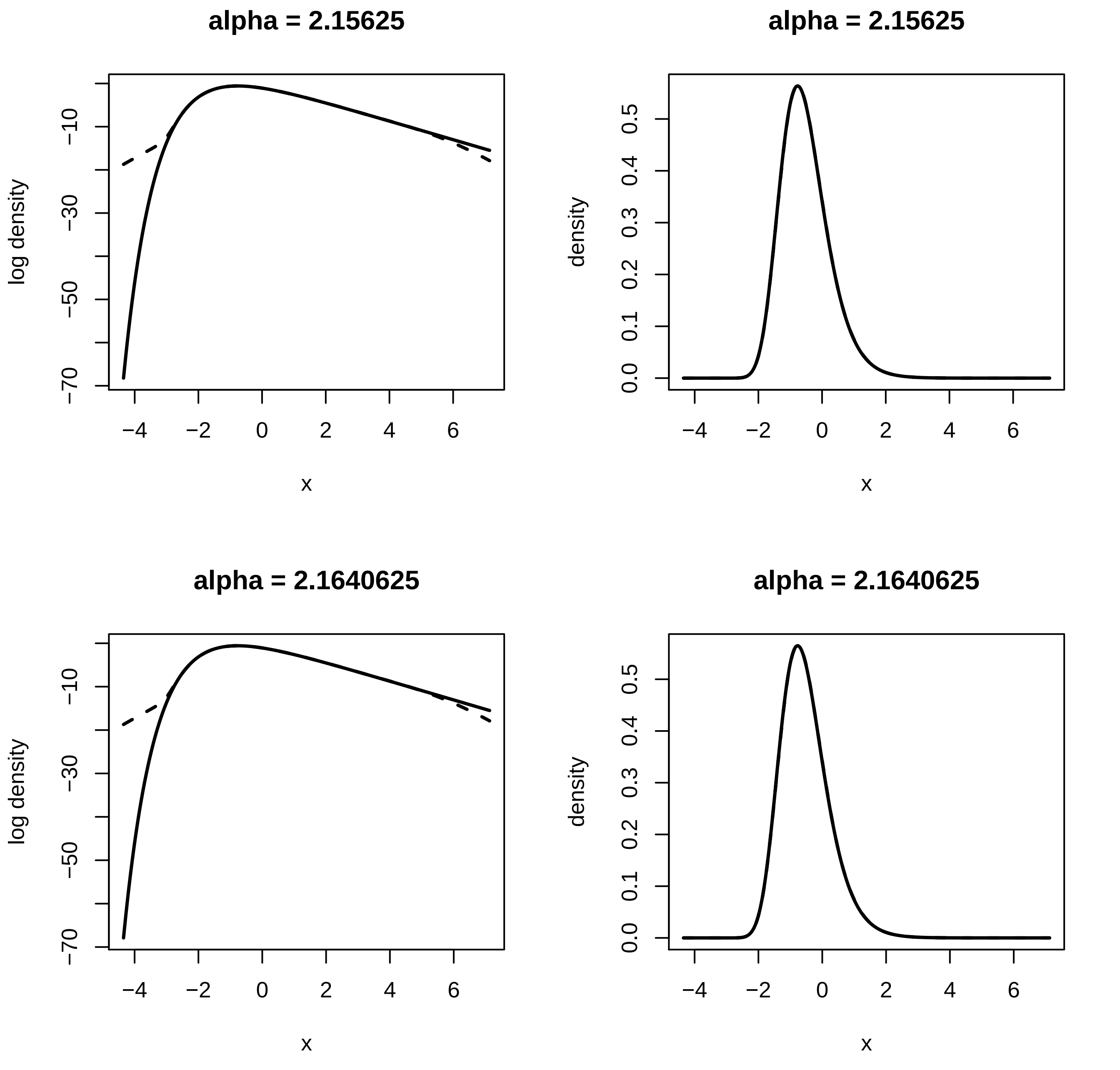

alpha = 2.09375
log density
x
alpha = 2.09375
density
x
alpha = 2.1015625
log density
x
alpha = 2.1015625
density
x

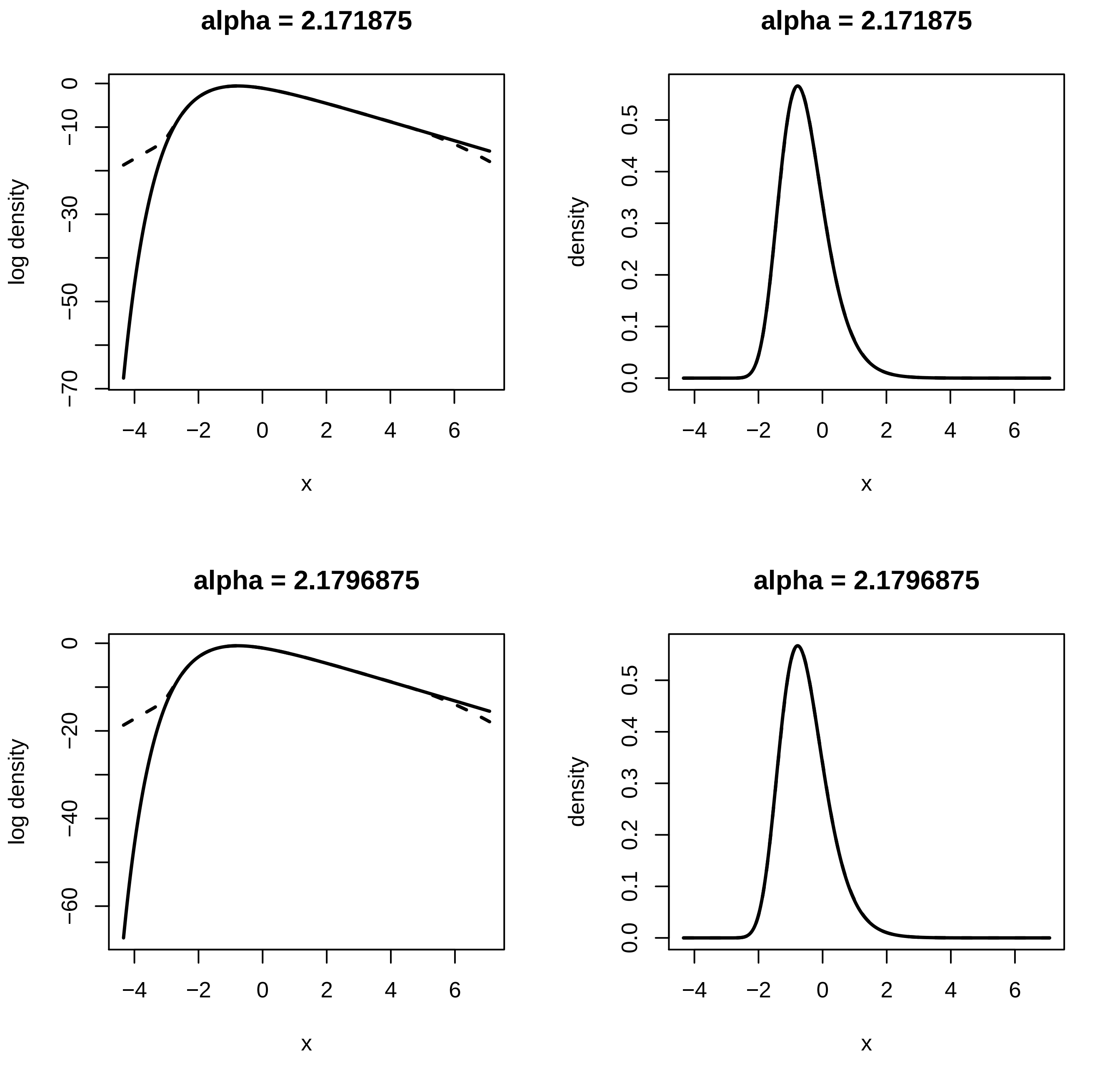

alpha = 2.109375
log density
−10
−30
−50
−70
−4
−2
0
2
4
6
x
alpha = 2.109375
density
0.0
0.1
0.2
0.3
0.4
0.5
−4
−2
0
2
4
6
x
alpha = 2.1171875
log density
−10
−30
−50
−70
−4
−2
0
2
4
6
x
alpha = 2.1171875
density
0.0
0.1
0.2
0.3
0.4
0.5
−4
−2
0
2
4
6
x

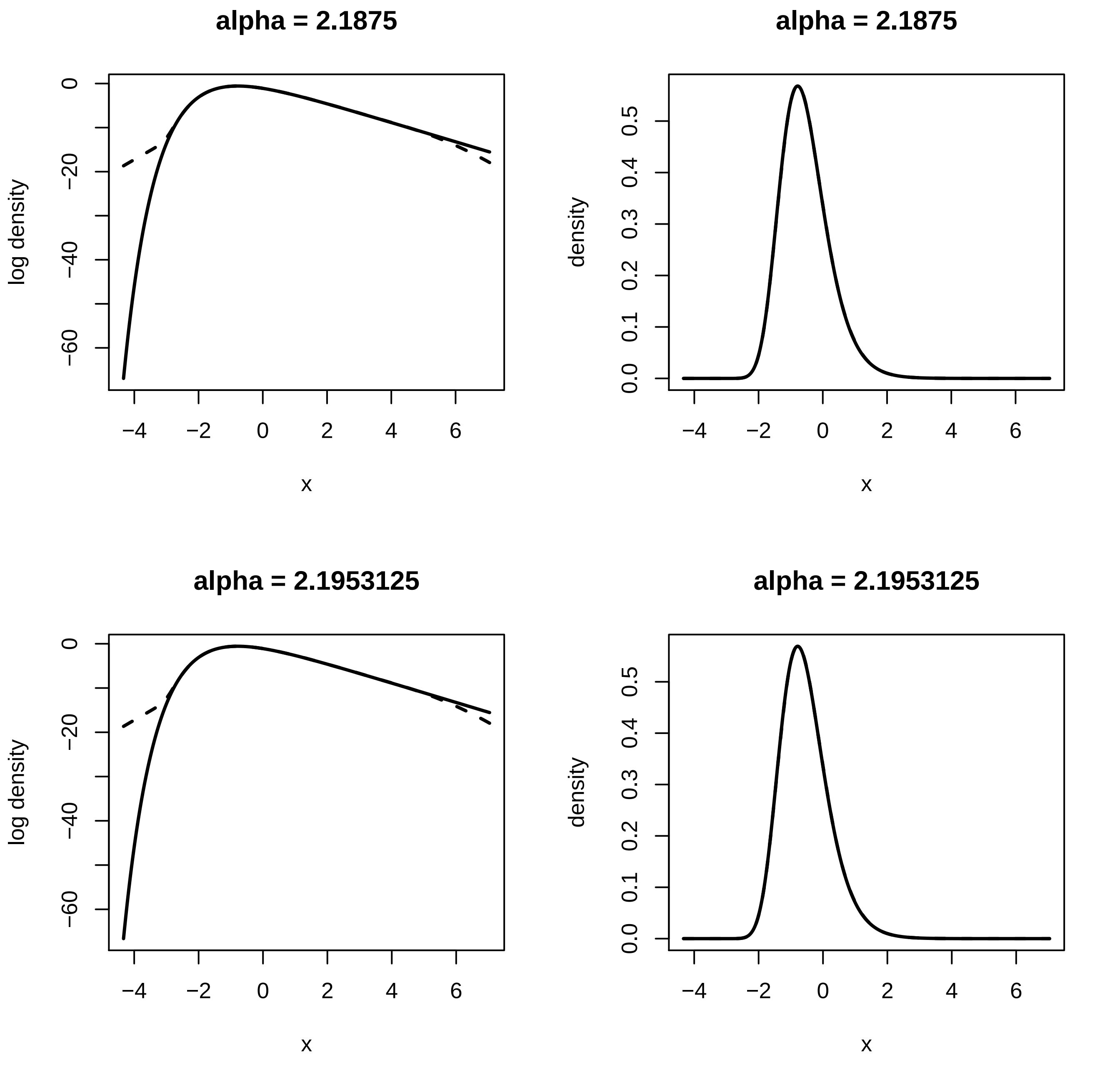

alpha = 2.125
log density
x
alpha = 2.125
density
x
alpha = 2.1328125
log density
x
alpha = 2.1328125
density
x

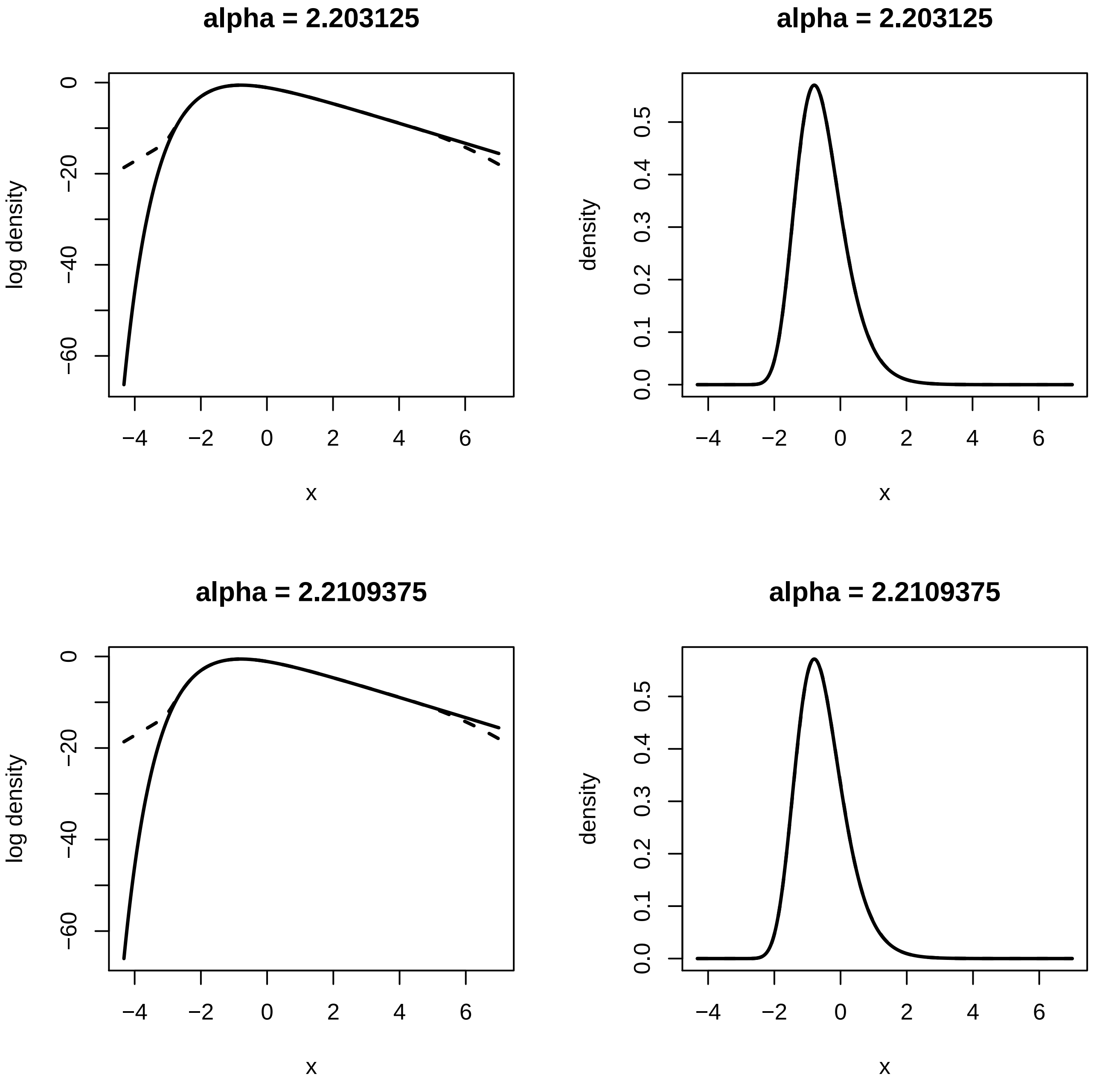


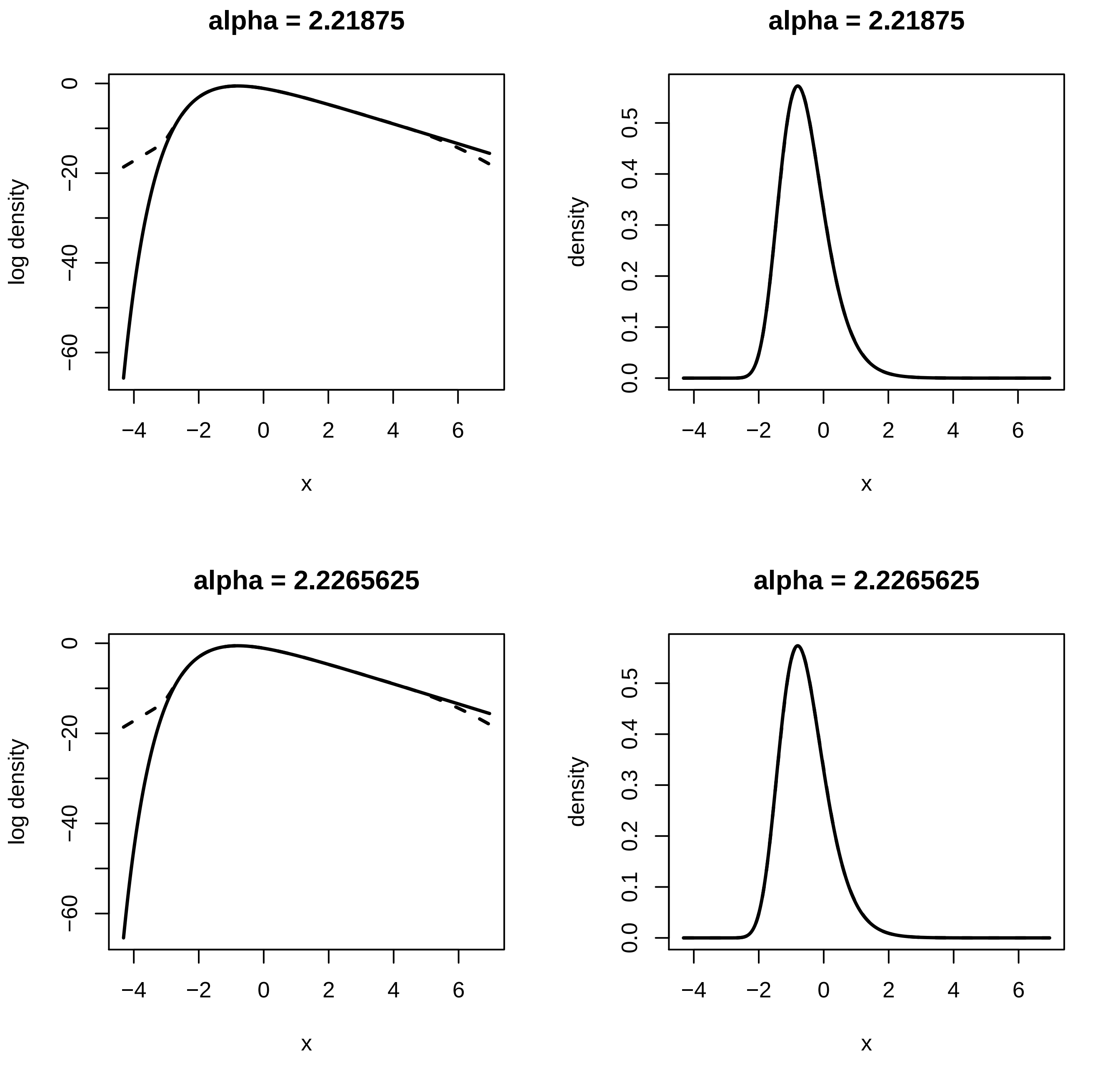
alpha = 2.140625
log density
−10
−30
−50
−70
−4
−2
0
2
4
6
x
alpha = 2.140625
density
0.0
0.1
0.2
0.3
0.4
0.5
−4
−2
0
2
4
6
x
alpha = 2.1484375
log density
−10
−30
−50
−70
−4
−2
0
2
4
6
x
alpha = 2.1484375
density
0.0
0.1
0.2
0.3
0.4
0.5
−4
−2
0
2
4
6
x

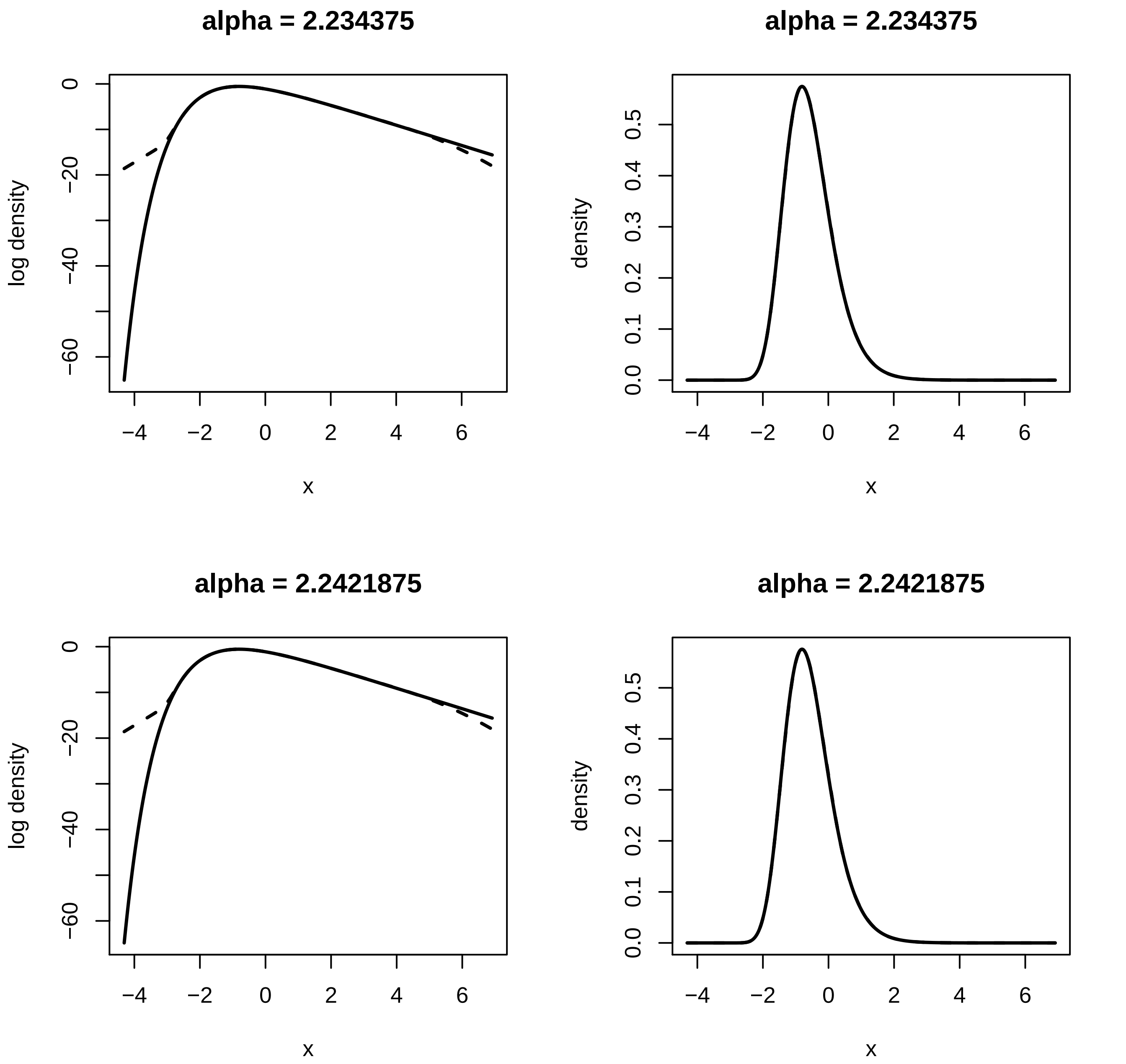

alpha = 2.15625
log density
x
alpha = 2.15625
density
x
alpha = 2.1640625
log density
x
alpha = 2.1640625
density
x


alpha = 2.171875
log density
0
−10
−30
−50
−70
−4
−2
0
2
4
6
x
alpha = 2.171875
density
0.0
0.1
0.2
0.3
0.4
0.5
−4
−2
0
2
4
6
x
alpha = 2.1796875
log density
0
−20
−40
−60
−4
−2
0
2
4
6
x
alpha = 2.1796875
density
0.0
0.1
0.2
0.3
0.4
0.5
−4
−2
0
2
4
6
x

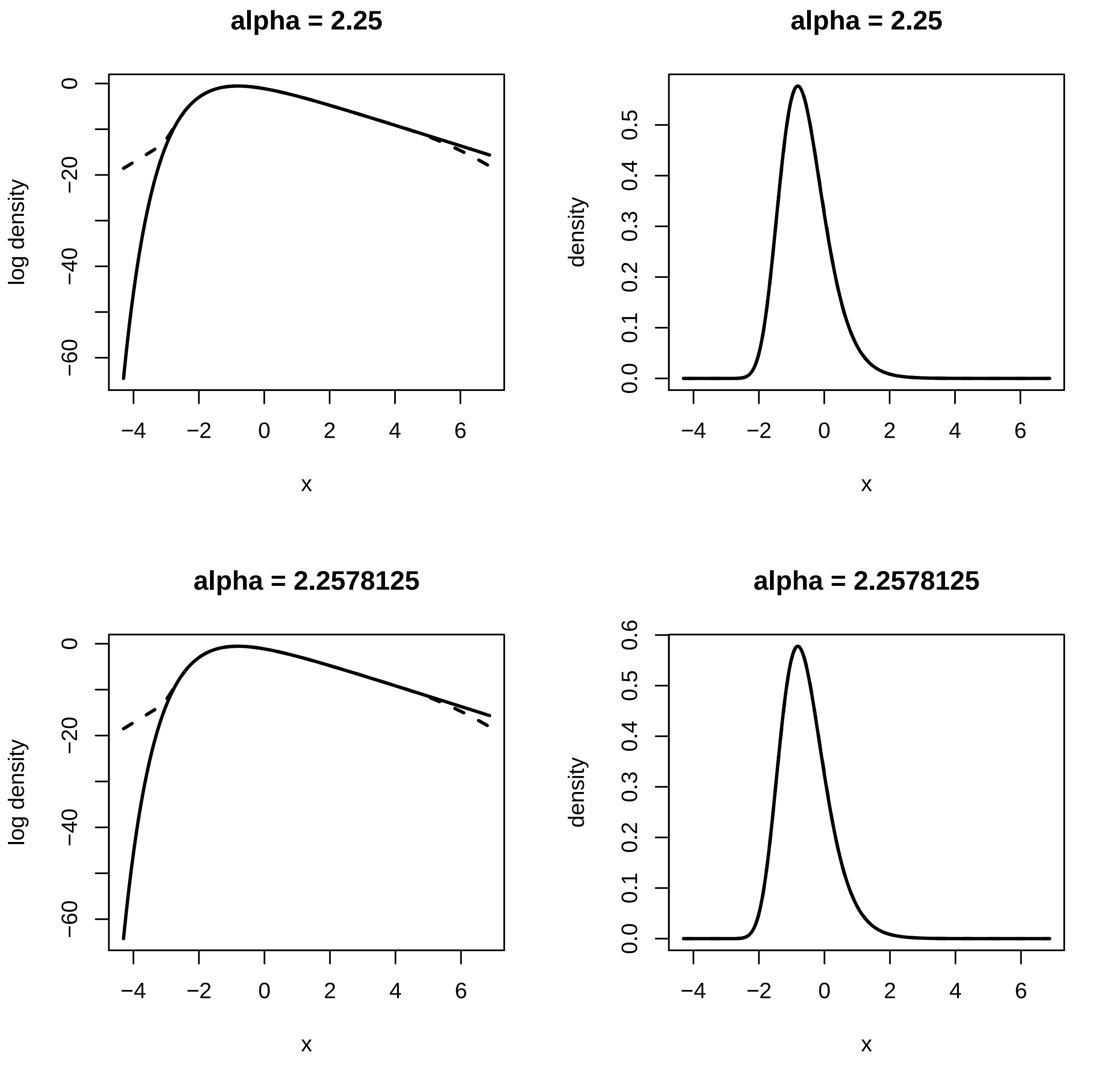

alpha = 2.1875
alpha = 2.1875
log density
density
x
x
alpha = 2.1953125
alpha = 2.1953125
log density
density
x
x
0
−20
−40
−60
0.0
0.1
0.2
0.3
0.4
0.5
−4
−2
0
2
4
6

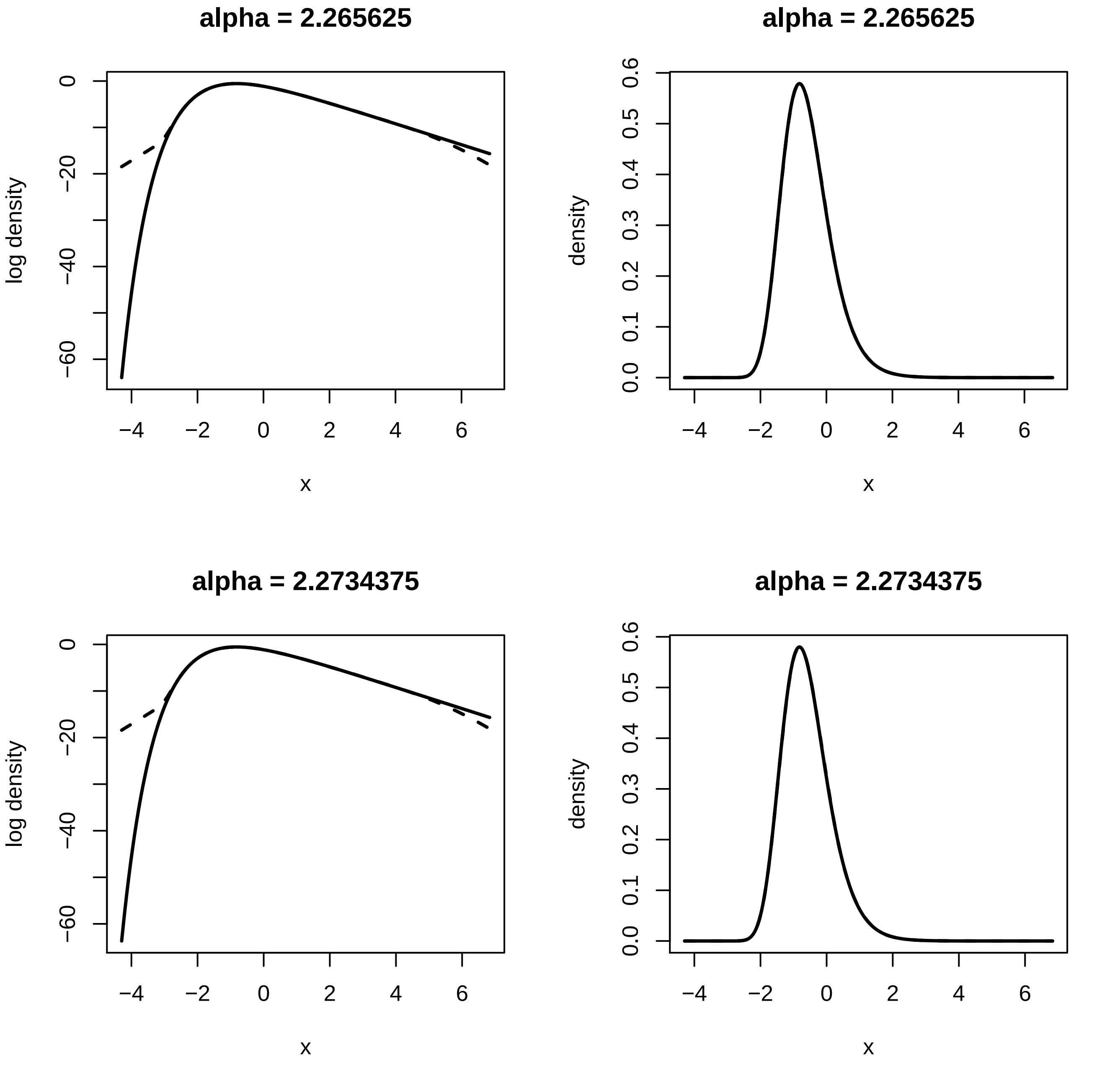

alpha = 2.203125
log density
0
−20
−40
−60
−4
−2
0
2
4
6
x
alpha = 2.203125
density
0.0
0.1
0.2
0.3
0.4
0.5
−4
−2
0
2
4
6
x
alpha = 2.2109375
log density
0
−20
−40
−60
−4
−2
0
2
4
6
x
alpha = 2.2109375
density
0.0
0.1
0.2
0.3
0.4
0.5
−4
−2
0
2
4
6
x

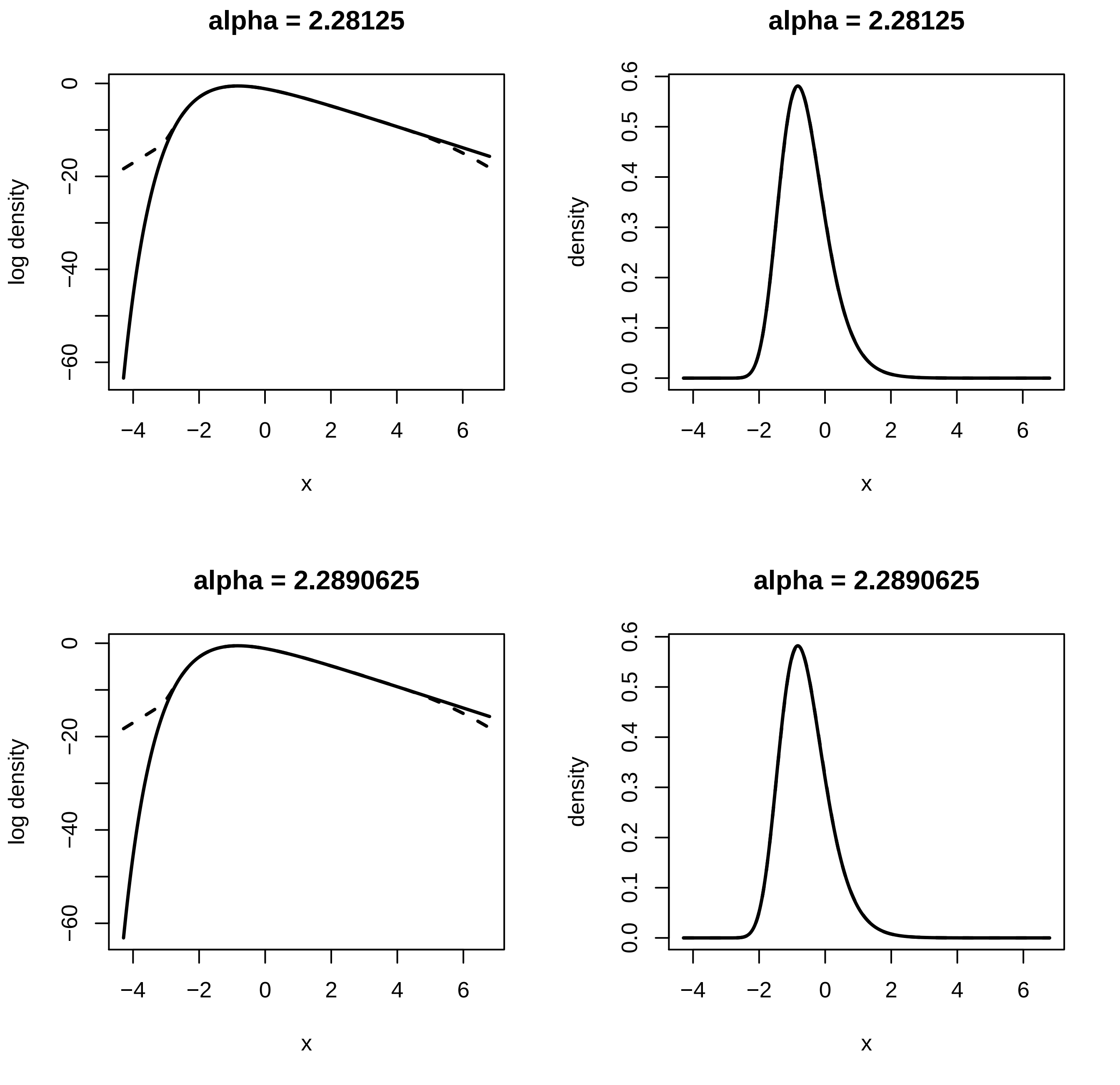

alpha = 2.21875
log density
0
−20
−40
−60
−4
−2
0
2
4
6
x
alpha = 2.21875
density
0.0
0.1
0.2
0.3
0.4
0.5
−4
−2
0
2
4
6
x
alpha = 2.2265625
log density
0
−20
−40
−60
−4
−2
0
2
4
6
x
alpha = 2.2265625
density
0.0
0.1
0.2
0.3
0.4
0.5
−4
−2
0
2
4
6
x



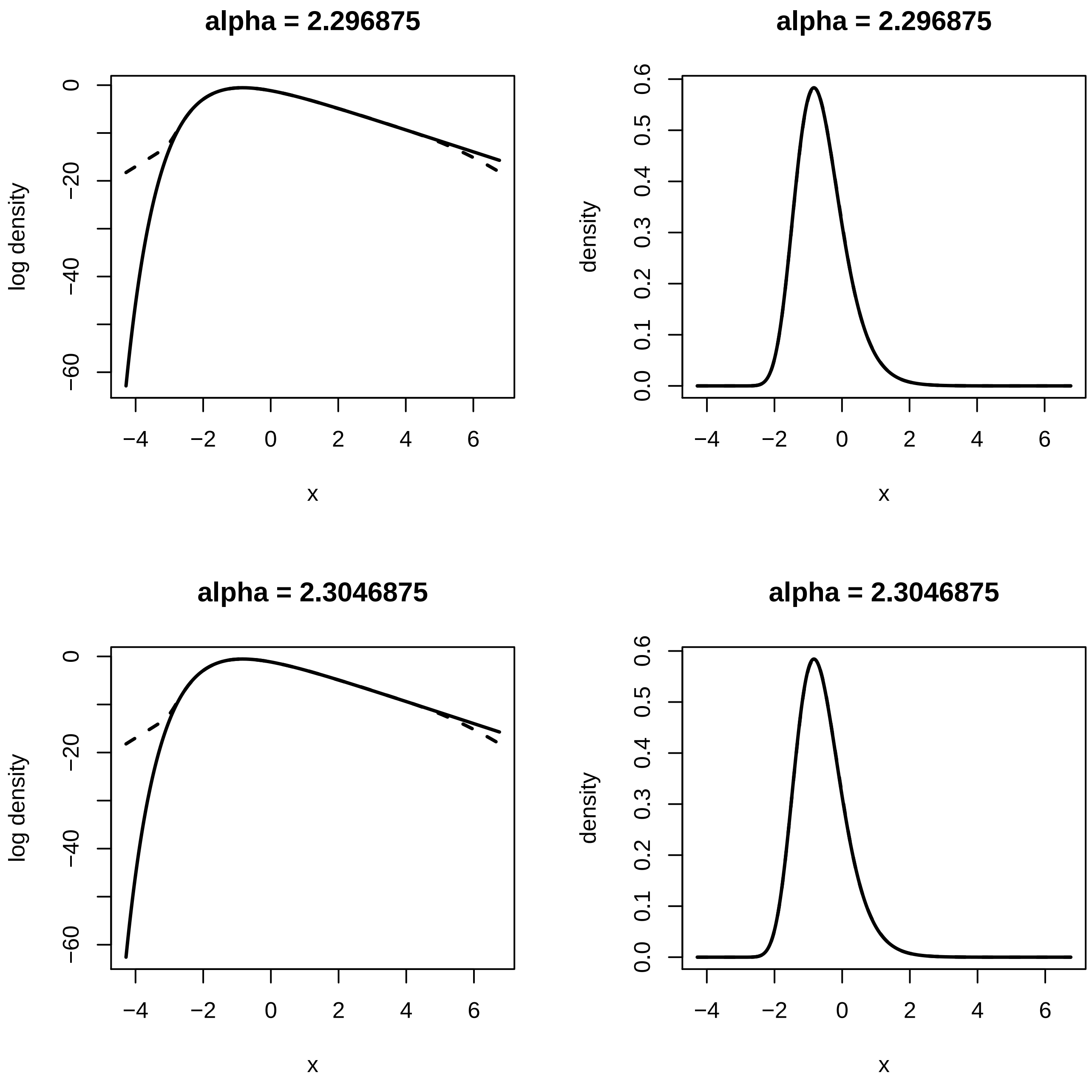
alpha = 2.234375
log density
0
−20
−40
−60
−4
−2
0
2
4
6
x
alpha = 2.234375
density
0.0
0.1
0.2
0.3
0.4
0.5
−4
−2
0
2
4
6
x
alpha = 2.2421875
log density
0
−20
−40
−60
−4
−2
0
2
4
6
x
alpha = 2.2421875
density
0.0
0.1
0.2
0.3
0.4
0.5
−4
−2
0
2
4
6
x

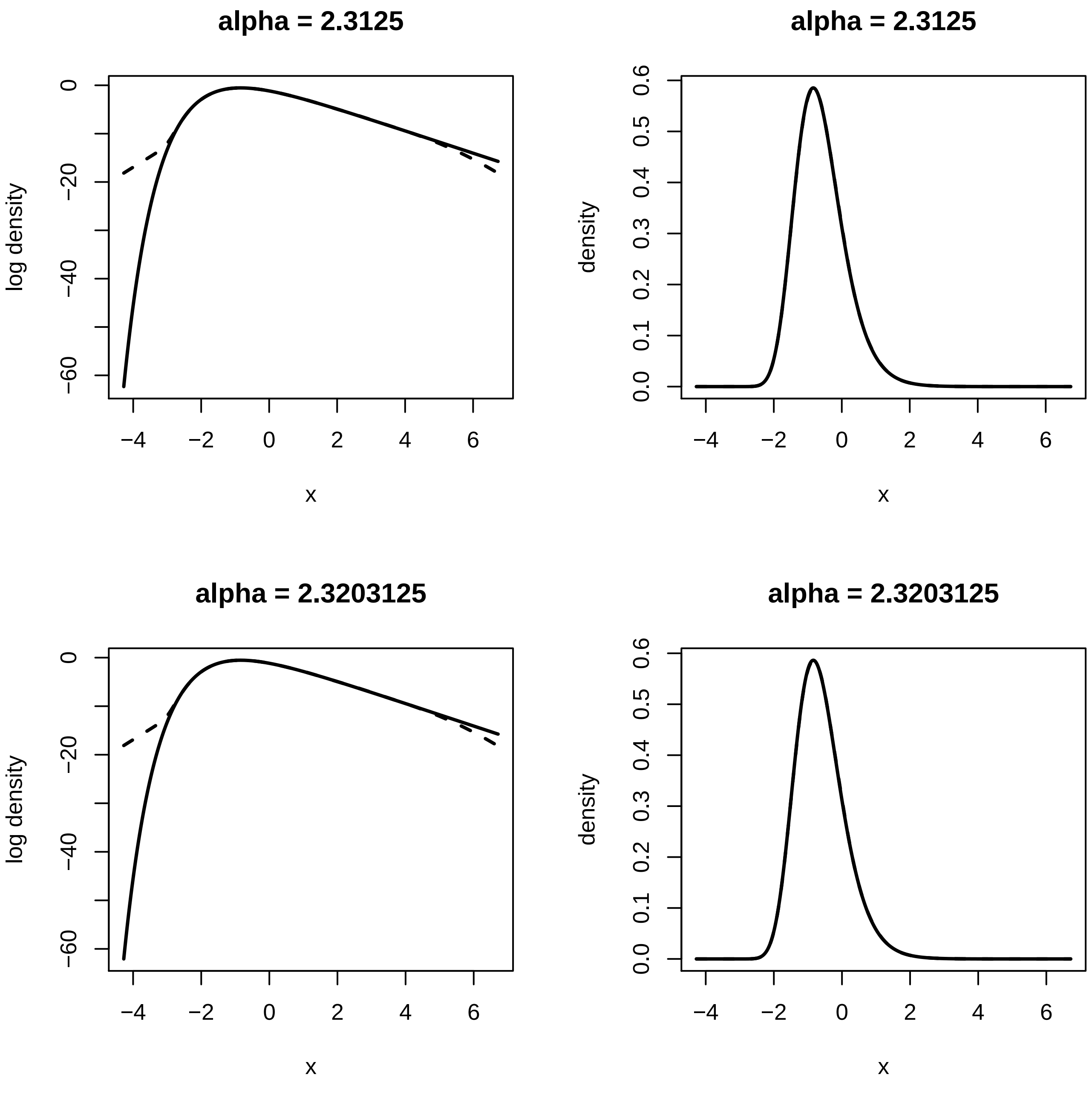

alpha = 2.25
log density
0
−20
−40
−60
−4
−2
0
2
4
6
x
alpha = 2.25
density
0.0
0.1
0.2
0.3
0.4
0.5
x
alpha = 2.2578125
log density
x
alpha = 2.2578125
density
0.6
x

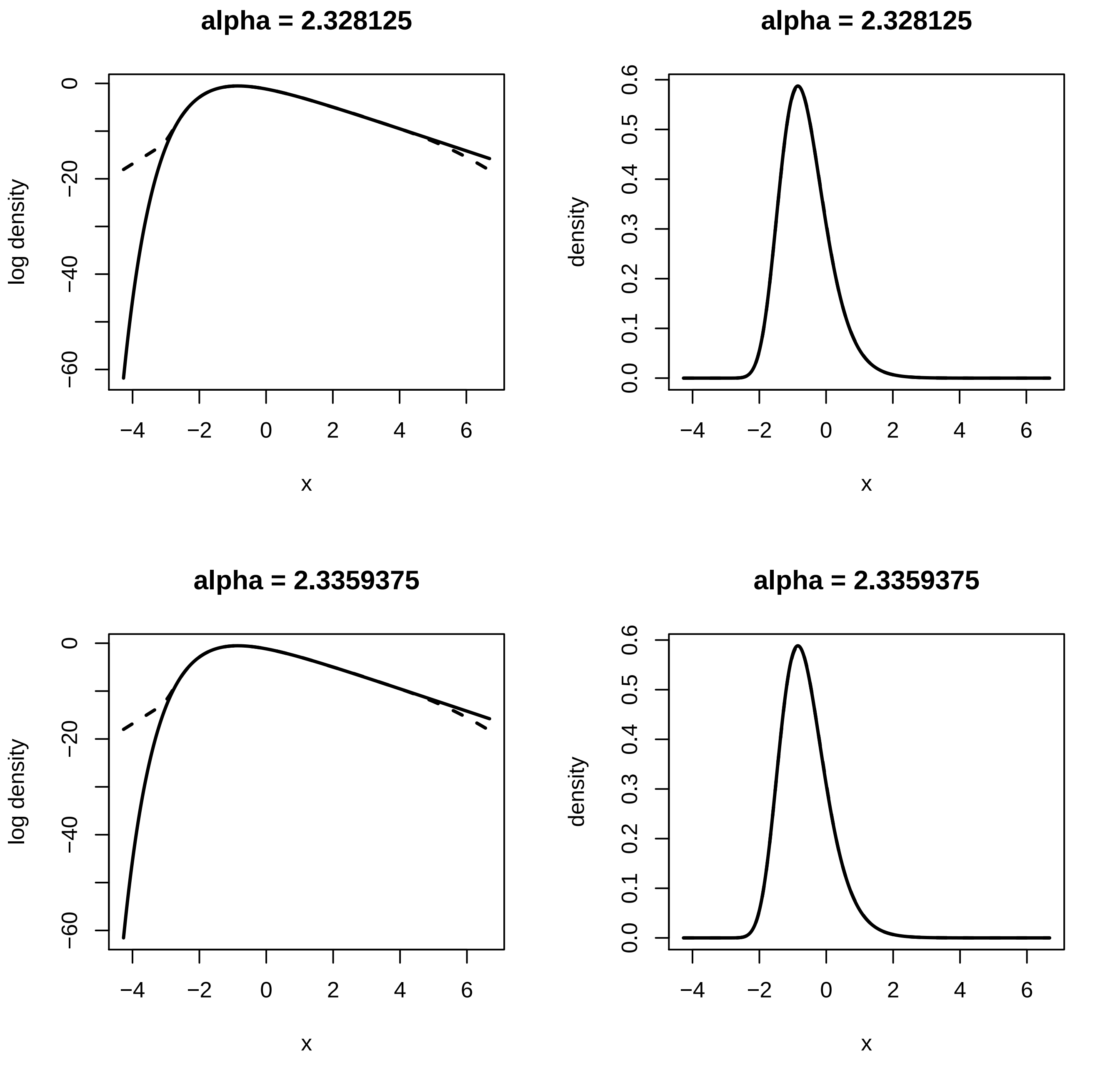

alpha = 2.265625
log density
x
alpha = 2.265625
density
x
alpha = 2.2734375
log density
x
alpha = 2.2734375
density
x

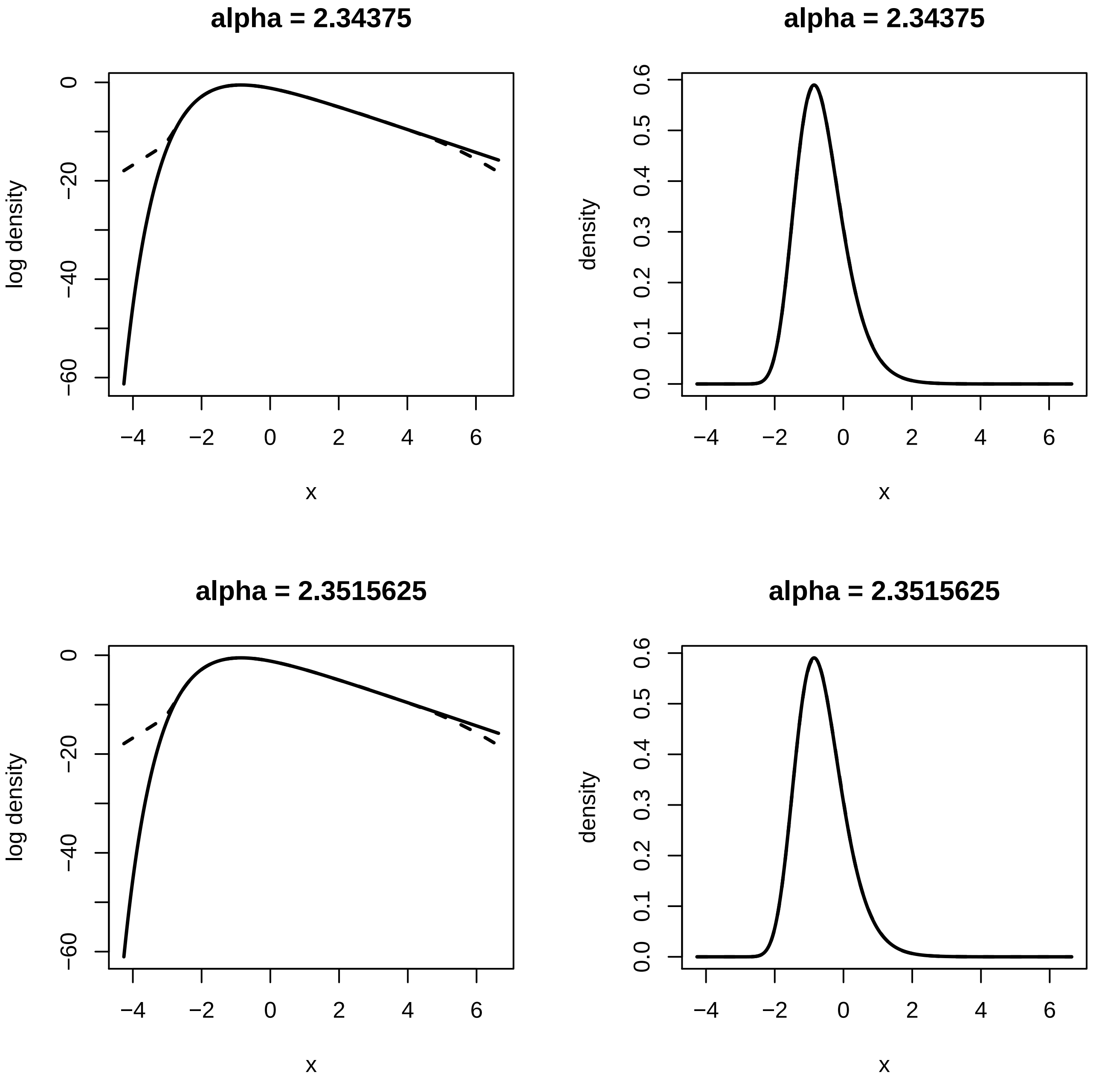


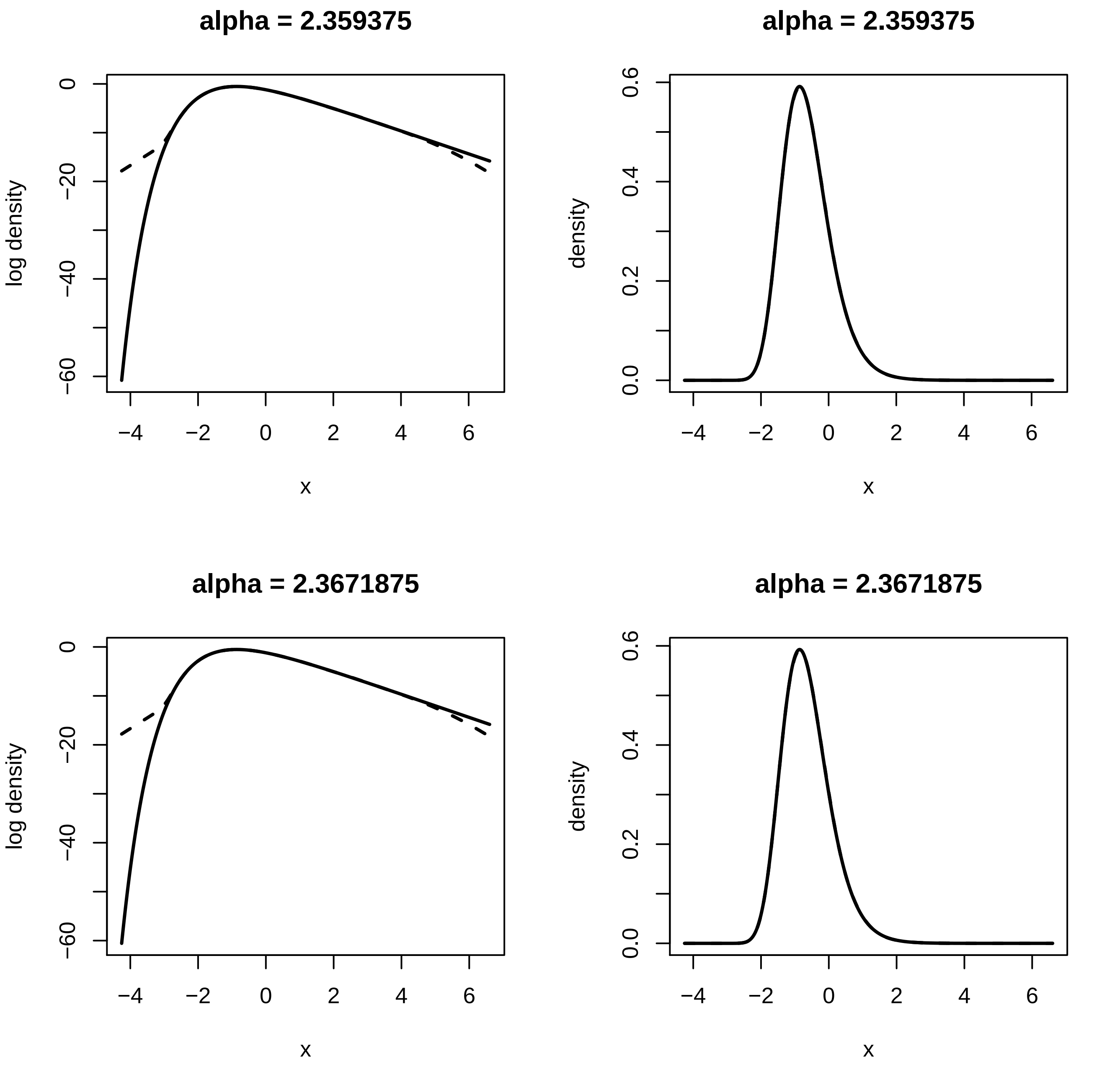
alpha = 2.28125
log density
0
−20
−40
−60
−4
−2
0
2
4
6
x
alpha = 2.28125
density
0.0
0.1
0.2
0.3
0.4
0.5
0.6
x
alpha = 2.2890625
log density
x
alpha = 2.2890625
density
x

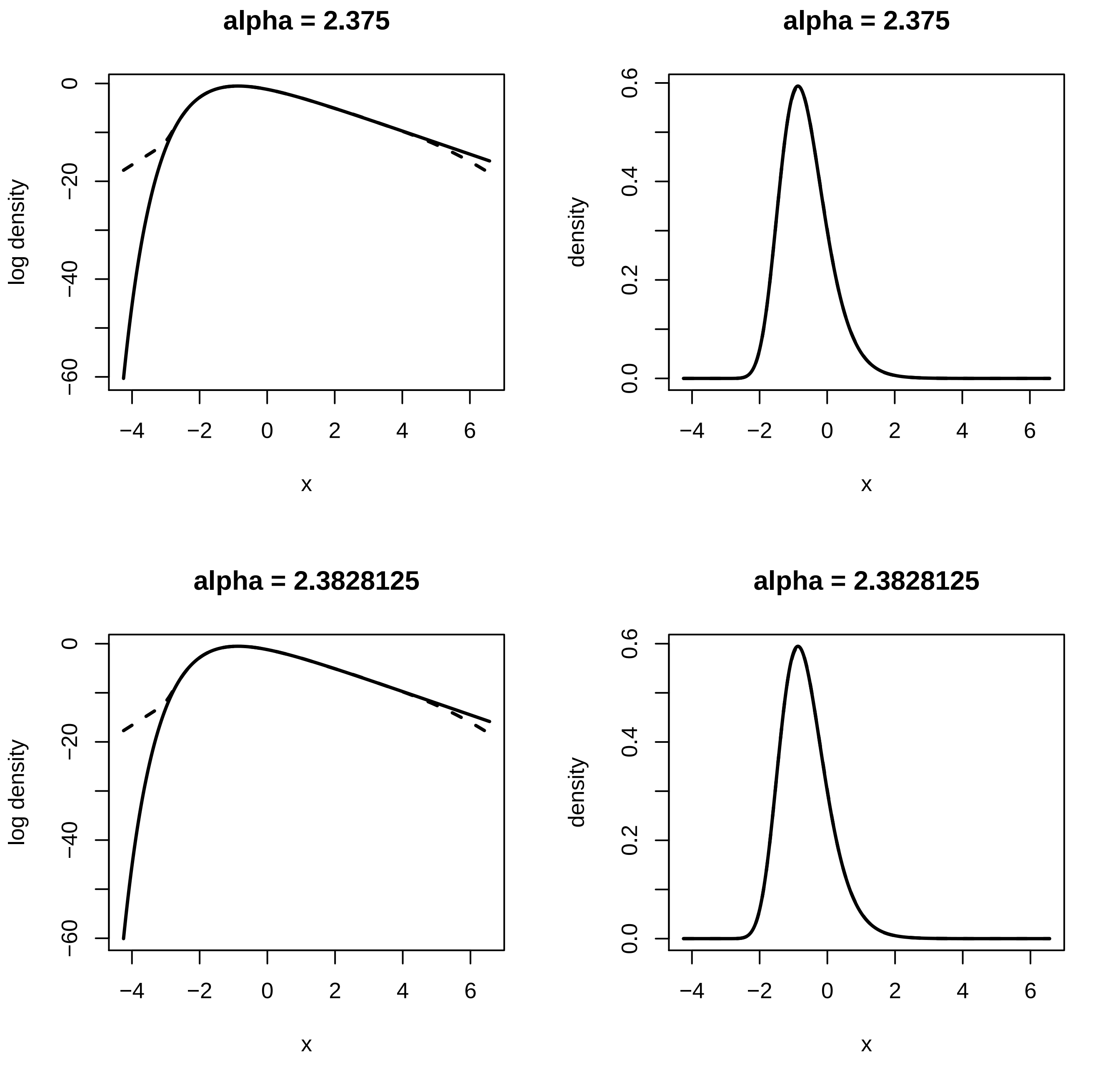

alpha = 2.296875
log density
x
alpha = 2.296875
density
x
alpha = 2.3046875
log density
x
alpha = 2.3046875
density
x

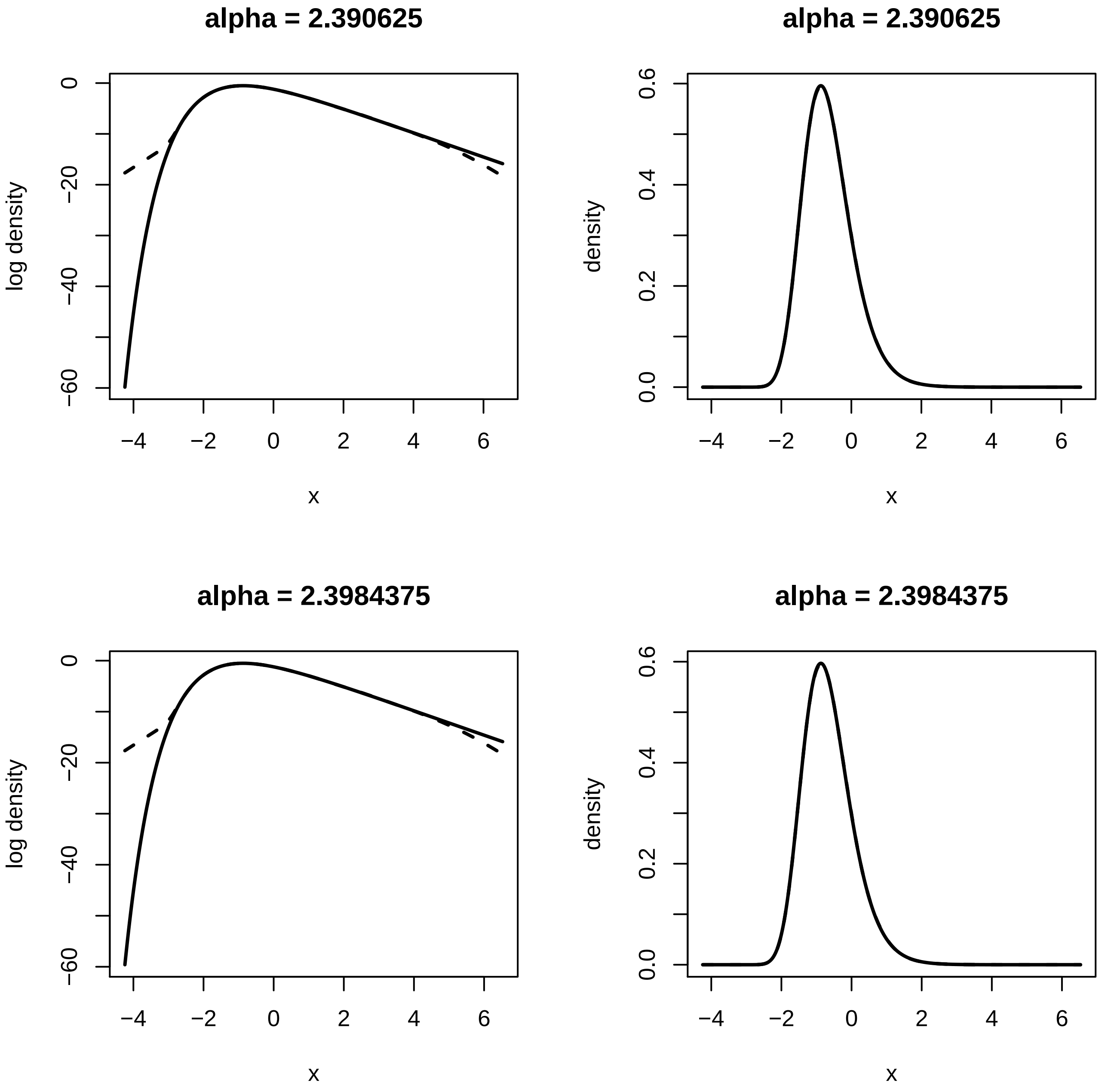

alpha = 2.3125
log density
x
0
−20
−40
−60
−4
−2
0
2
4
6
alpha = 2.3125
density
x
0.0
0.1
0.2
0.3
0.4
0.5
0.6
−4
−2
0
2
4
6
alpha = 2.3203125
log density
x
0
−20
−40
−60
−4
−2
0
2
4
6
alpha = 2.3203125
density
x
0.0
0.1
0.2
0.3
0.4
0.5
0.6
−4
−2
0
2
4
6

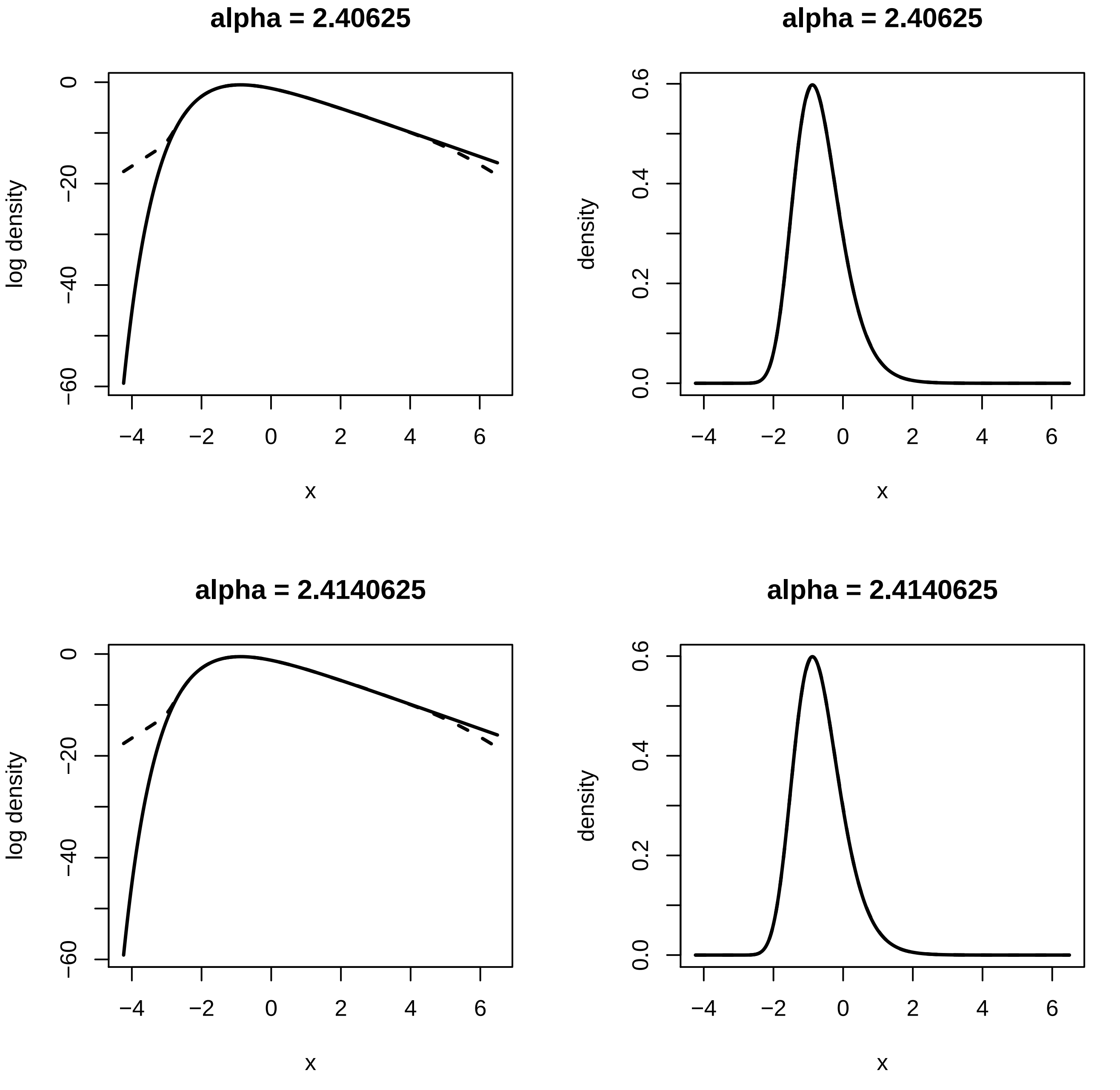

alpha = 2.328125
log density
0
−20
−40
−60
−4
−2
0
2
4
6
x
alpha = 2.328125
density
0.0
0.1
0.2
0.3
0.4
0.5
0.6
−4
−2
0
2
4
6
x
alpha = 2.3359375
log density
0
−20
−40
−60
−4
−2
0
2
4
6
x
alpha = 2.3359375
density
0.0
0.1
0.2
0.3
0.4
0.5
0.6
−4
−2
0
2
4
6
x

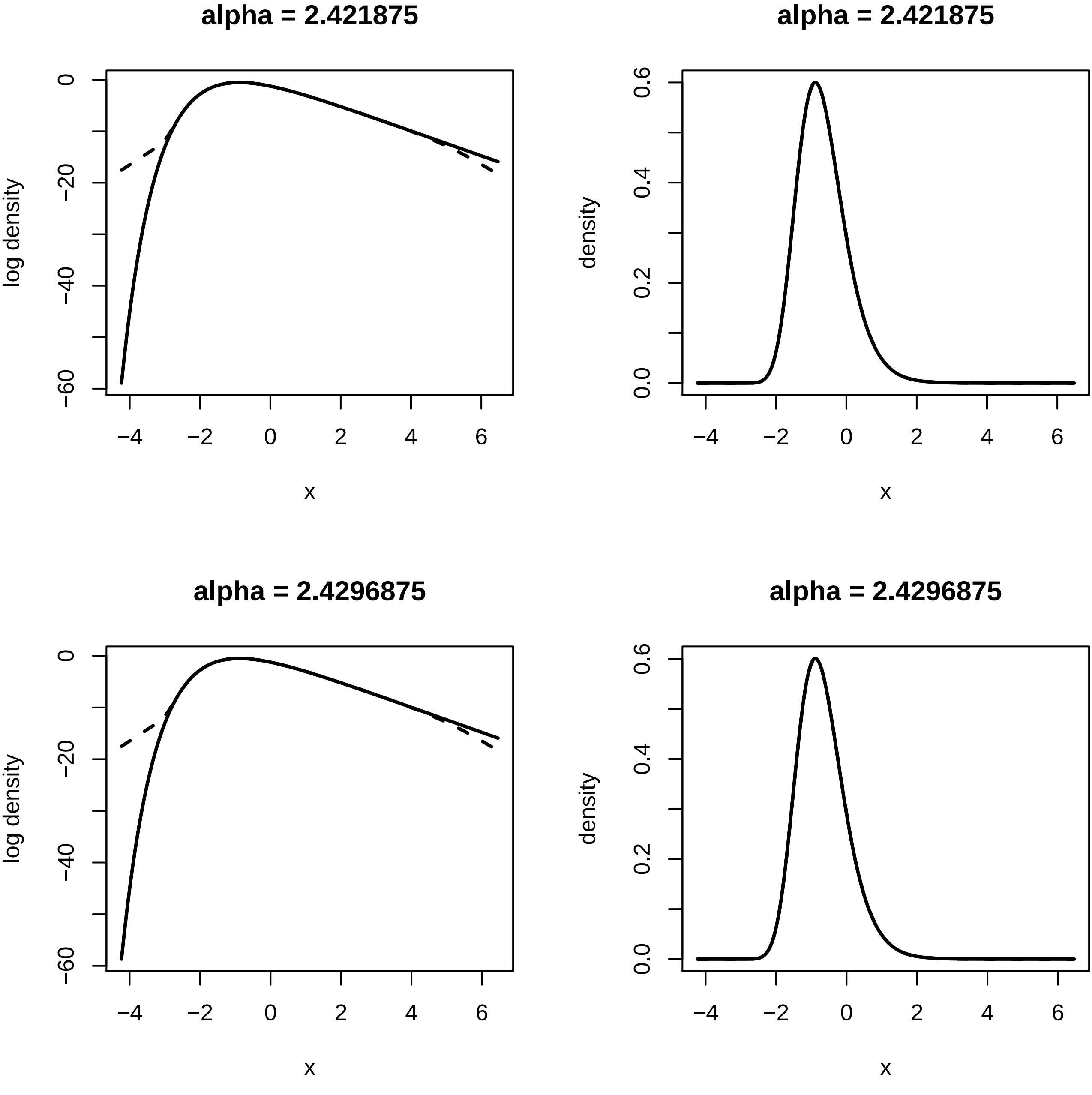

alpha = 2.34375
log density
x
density
alpha = 2.3515625
0
−20
−40
−60
−4
−2
2
4
6
0.0
0.1
0.2
0.3
0.4
0.5
0.6

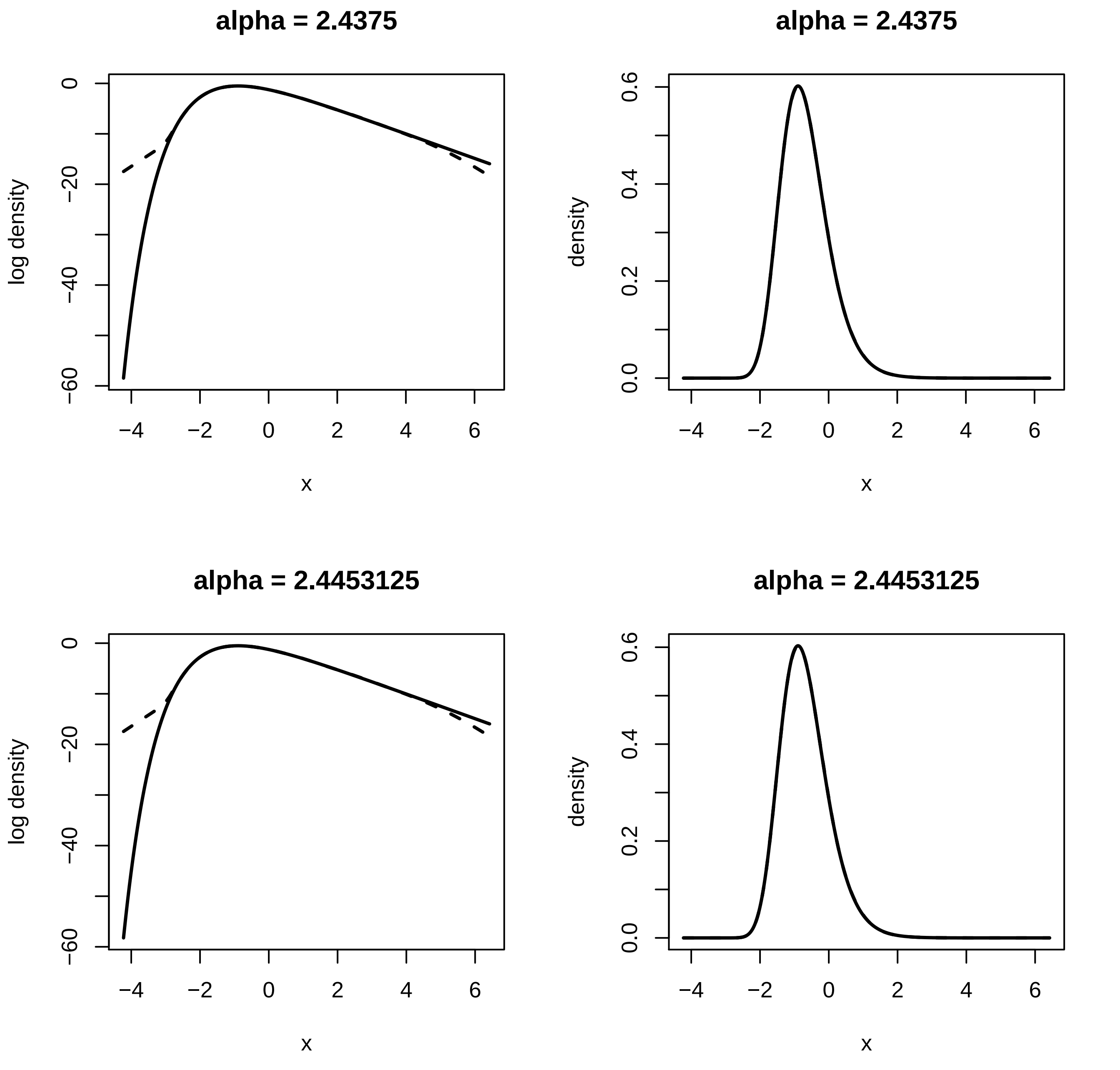

alpha = 2.359375
log density
0
−20
−40
−60
−4
−2
0
2
4
6
x
alpha = 2.359375
density
0.0
0.2
0.4
0.6
−4
−2
0
2
4
6
x
alpha = 2.3671875
log density
0
−20
−40
−60
−4
−2
0
2
4
6
x
alpha = 2.3671875
density
0.0
0.2
0.4
0.6
−4
−2
0
2
4
6
x


alpha = 2.375
log density
0
−20
−40
−60
−4
−2
0
2
4
6
x
alpha = 2.375
density
0.6
0.4
0.2
0.0
−4
−2
0
2
4
6
x
alpha = 2.3828125
log density
0
−20
−40
−60
−4
−2
0
2
4
6
x
alpha = 2.3828125
density
0.6
0.4
0.2
0.0
−4
−2
0
2
4
6
x

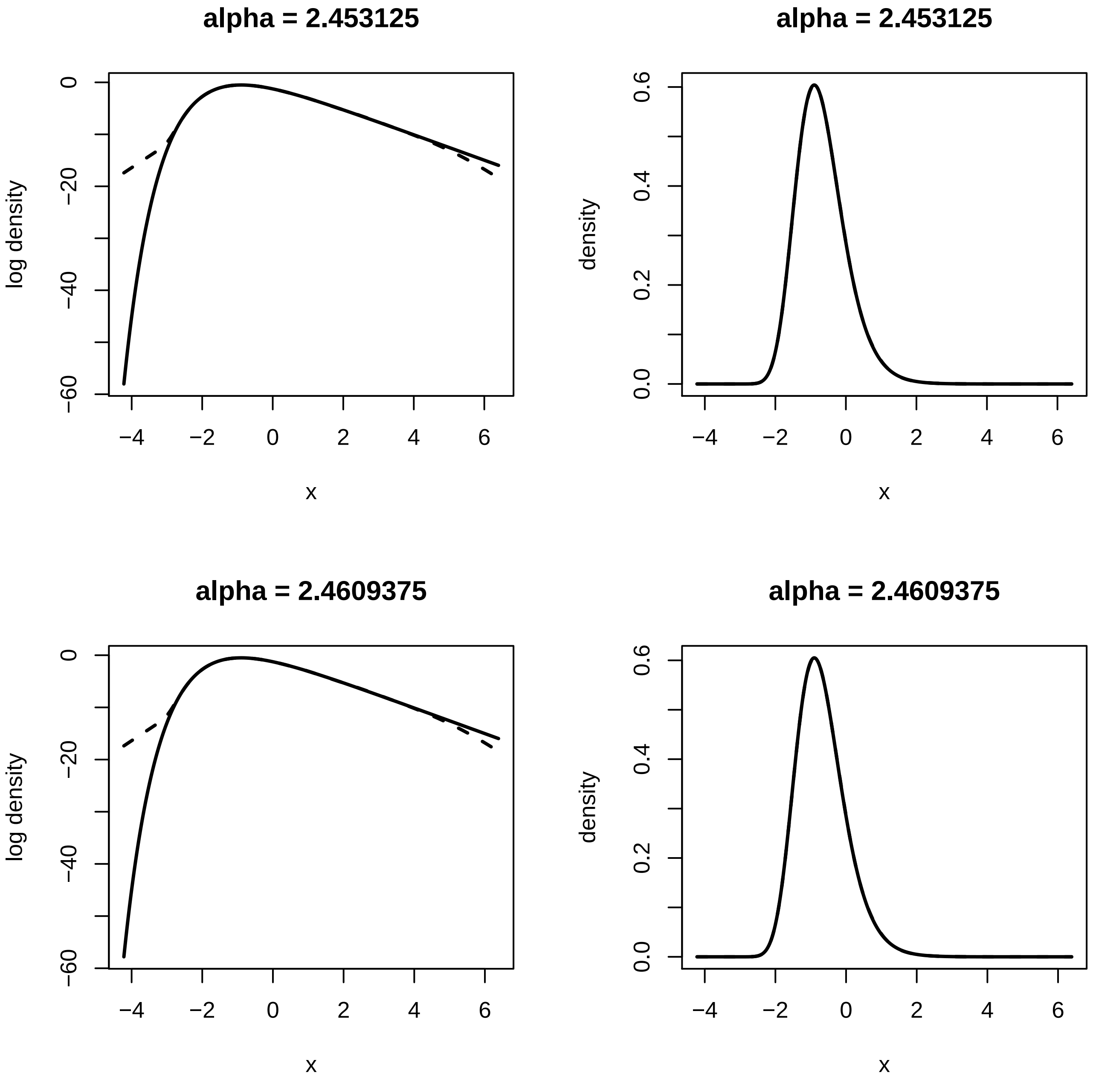

alpha = 2.390625
log density
0
−20
−40
−60
−4
−2
0
2
4
6
x
alpha = 2.390625
density
0.6
0.4
0.2
0.0
x
alpha = 2.3984375
log density
x
alpha = 2.3984375
density
x

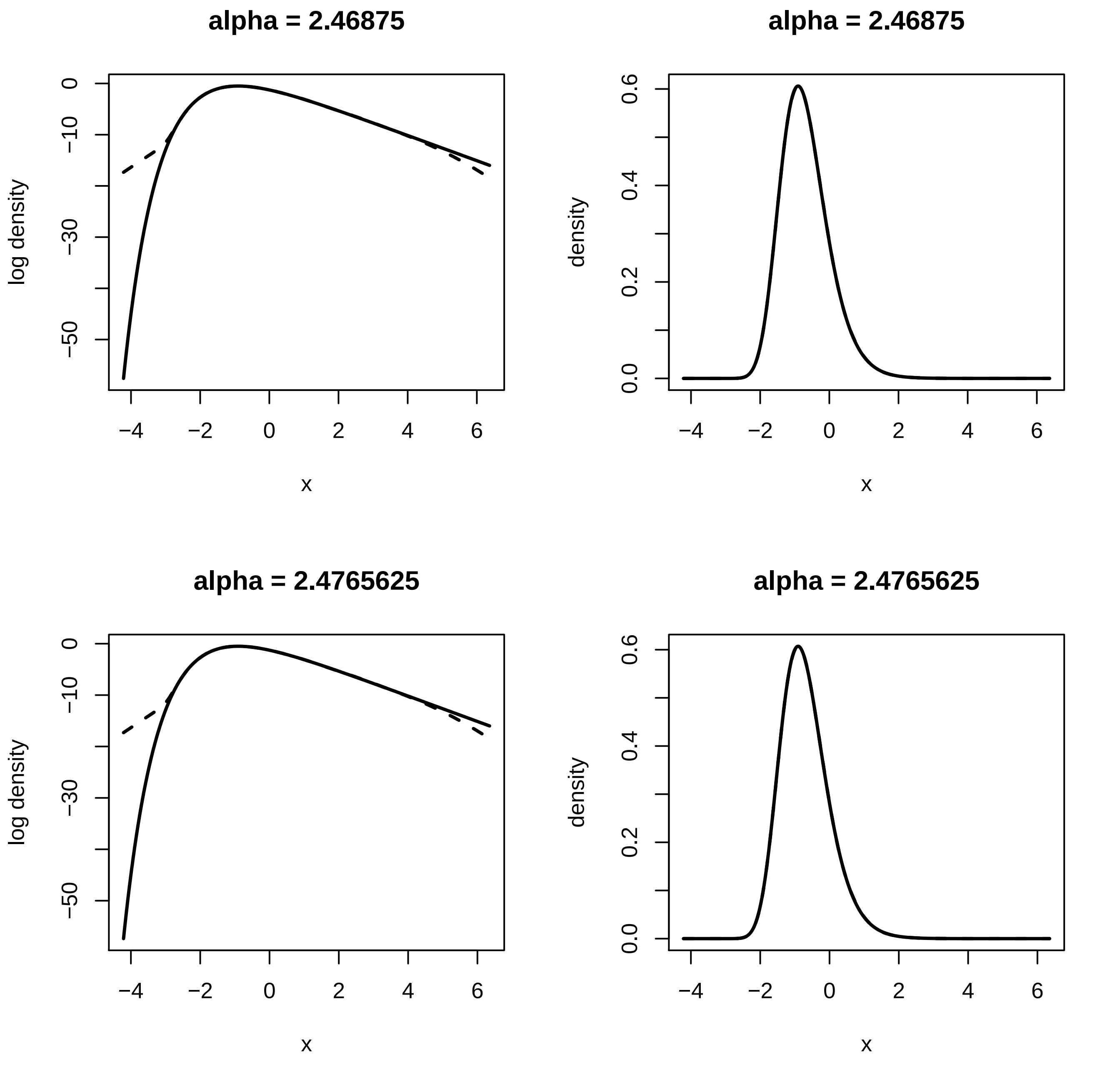

alpha = 2.40625
log density
0
−20
−40
−60
−4
−2
0
2
4
6
x
alpha = 2.40625
density
0.0
0.2
0.4
0.6
−4
−2
0
2
4
6
x
alpha = 2.4140625
log density
0
−20
−40
−60
−4
−2
0
2
4
6
x
alpha = 2.4140625
density
0.0
0.2
0.4
0.6
−4
−2
0
2
4
6
x


alpha = 2.421875
log density
0
−20
−40
−60
−4
−2
0
2
4
6
x
alpha = 2.421875
density
0.0
0.2
0.4
0.6
−4
−2
0
2
4
6
x
alpha = 2.4296875
log density
0
−20
−40
−60
−4
−2
0
2
4
6
x
alpha = 2.4296875
density
0.0
0.2
0.4
0.6
−4
−2
0
2
4
6
x

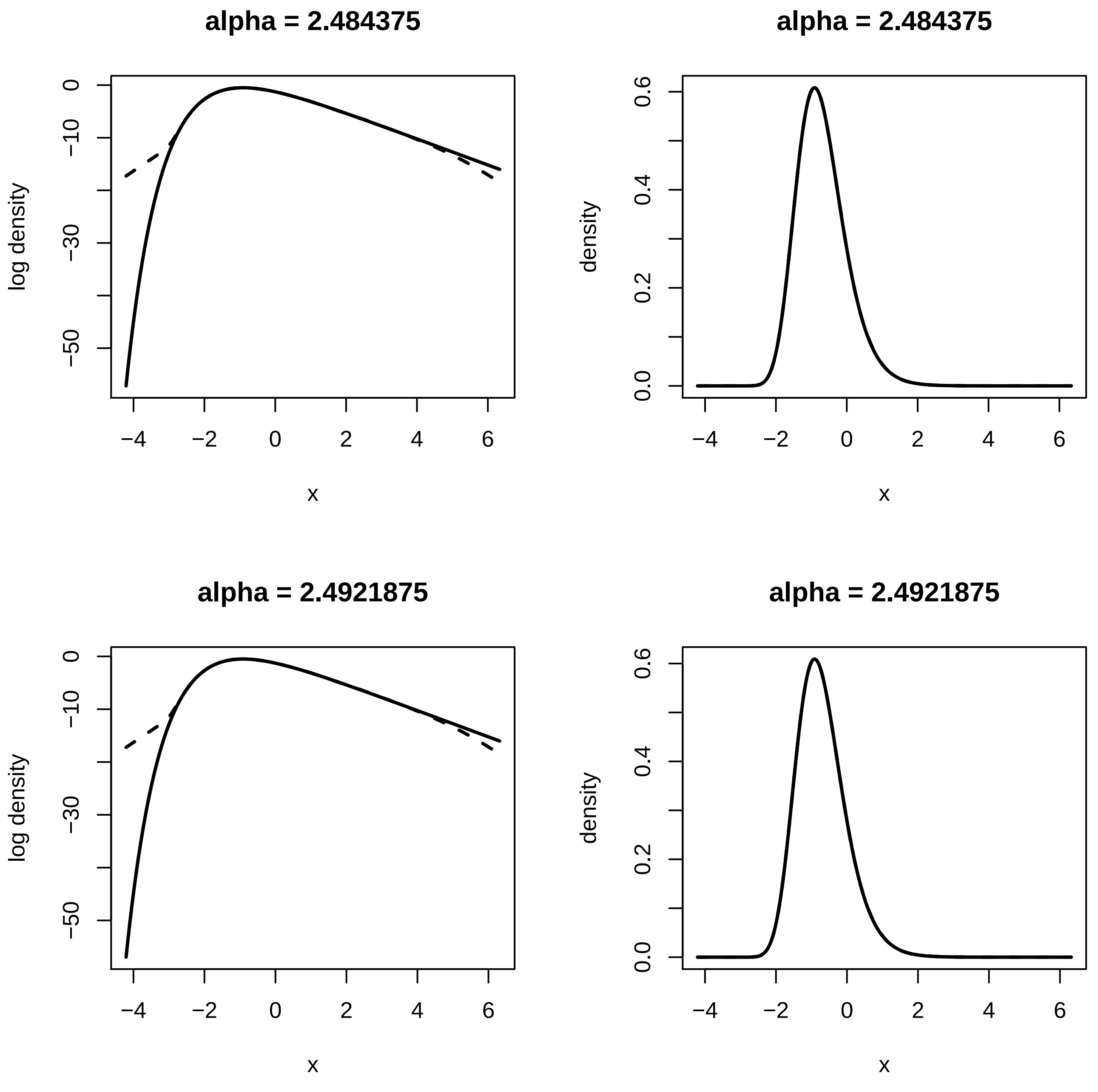


alpha = 2.4375
log density
x
alpha = 2.4375
density
x
alpha = 2.4453125
log density
x
alpha = 2.4453125
density
x

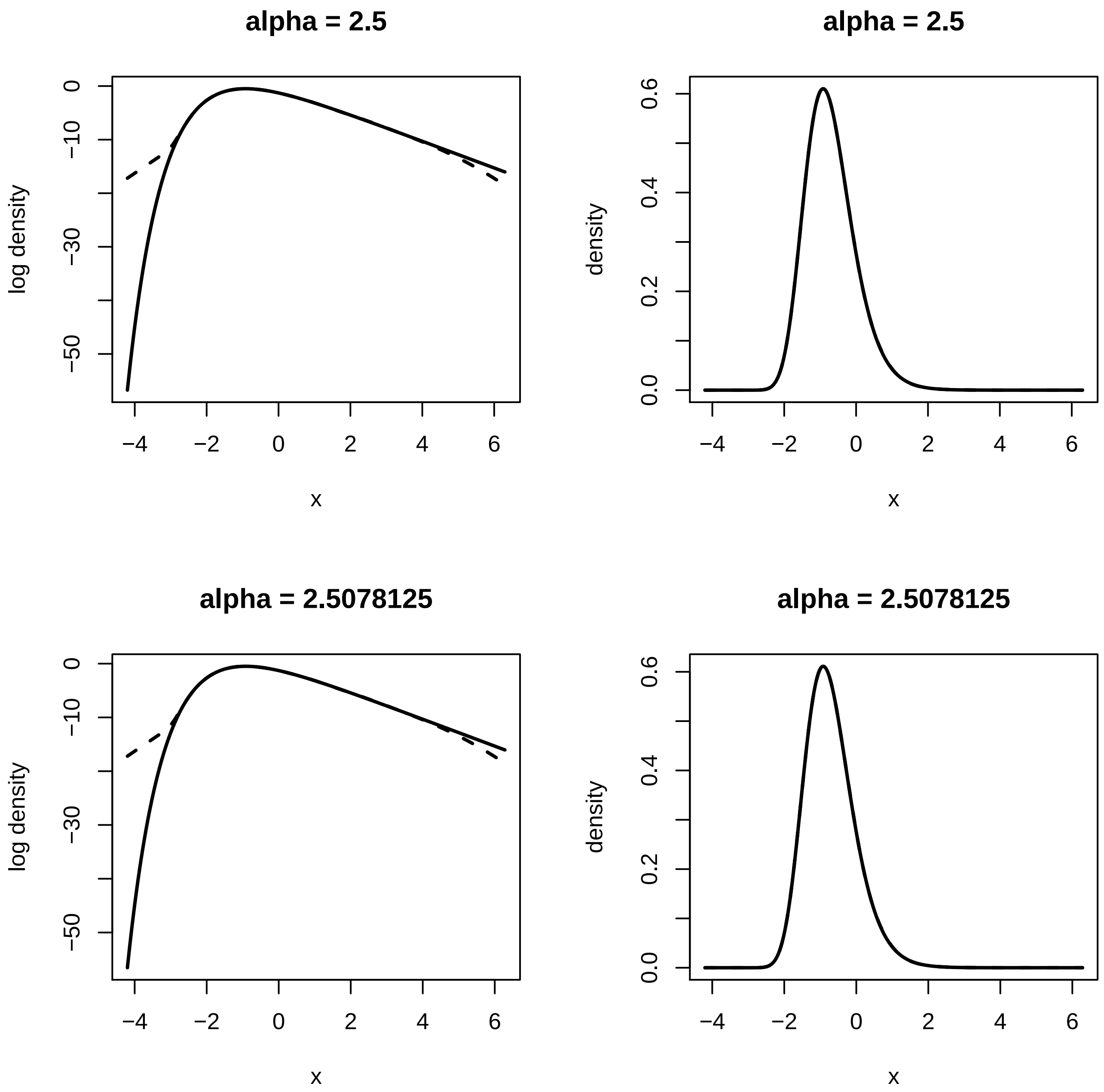

alpha = 2.453125
log density
0
−20
−40
−60
−4
−2
0
2
4
6
x
alpha = 2.453125
density
0.0
0.2
0.4
0.6
x
alpha = 2.4609375
log density
x
alpha = 2.4609375
density
x

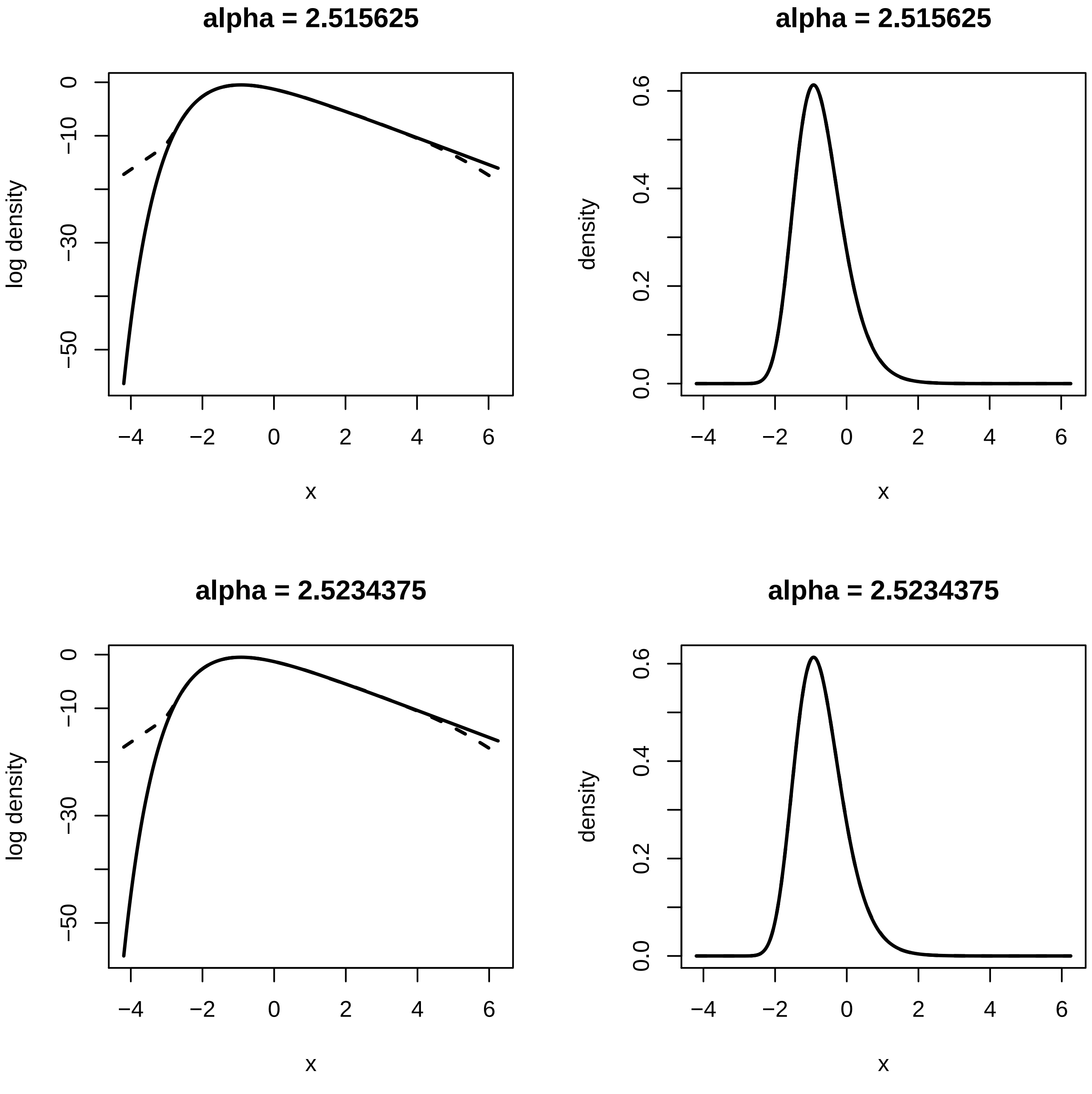


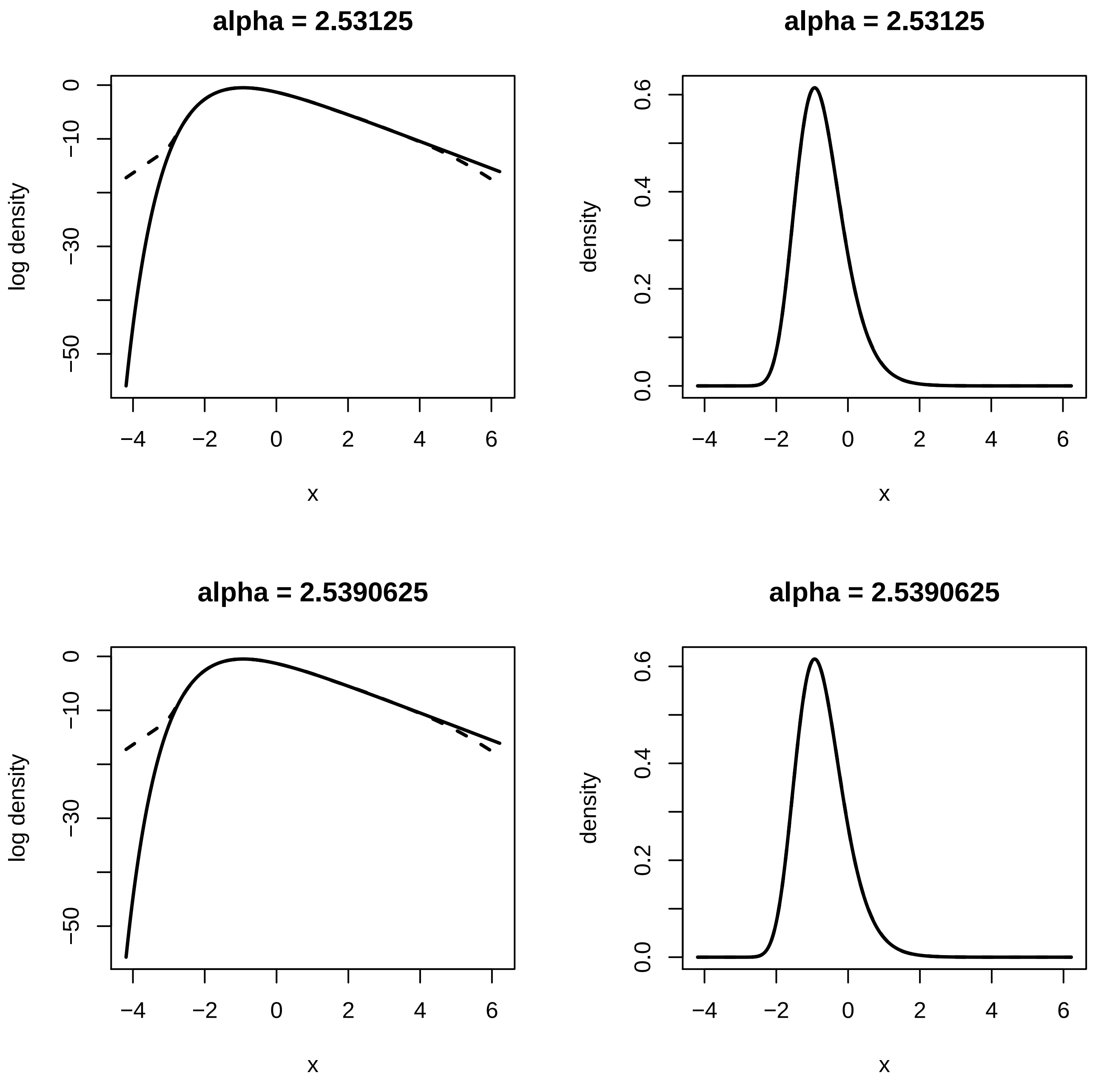
alpha = 2.46875
log density
0
−10
−30
−50
−4
−2
0
2
4
6
x
alpha = 2.46875
density
0.6
0.4
0.2
0.0
−4
−2
0
2
4
6
x
alpha = 2.4765625
log density
0
−10
−30
−50
−4
−2
0
2
4
6
x
alpha = 2.4765625
density
0.6
0.4
0.2
0.0
−4
−2
0
2
4
6
x

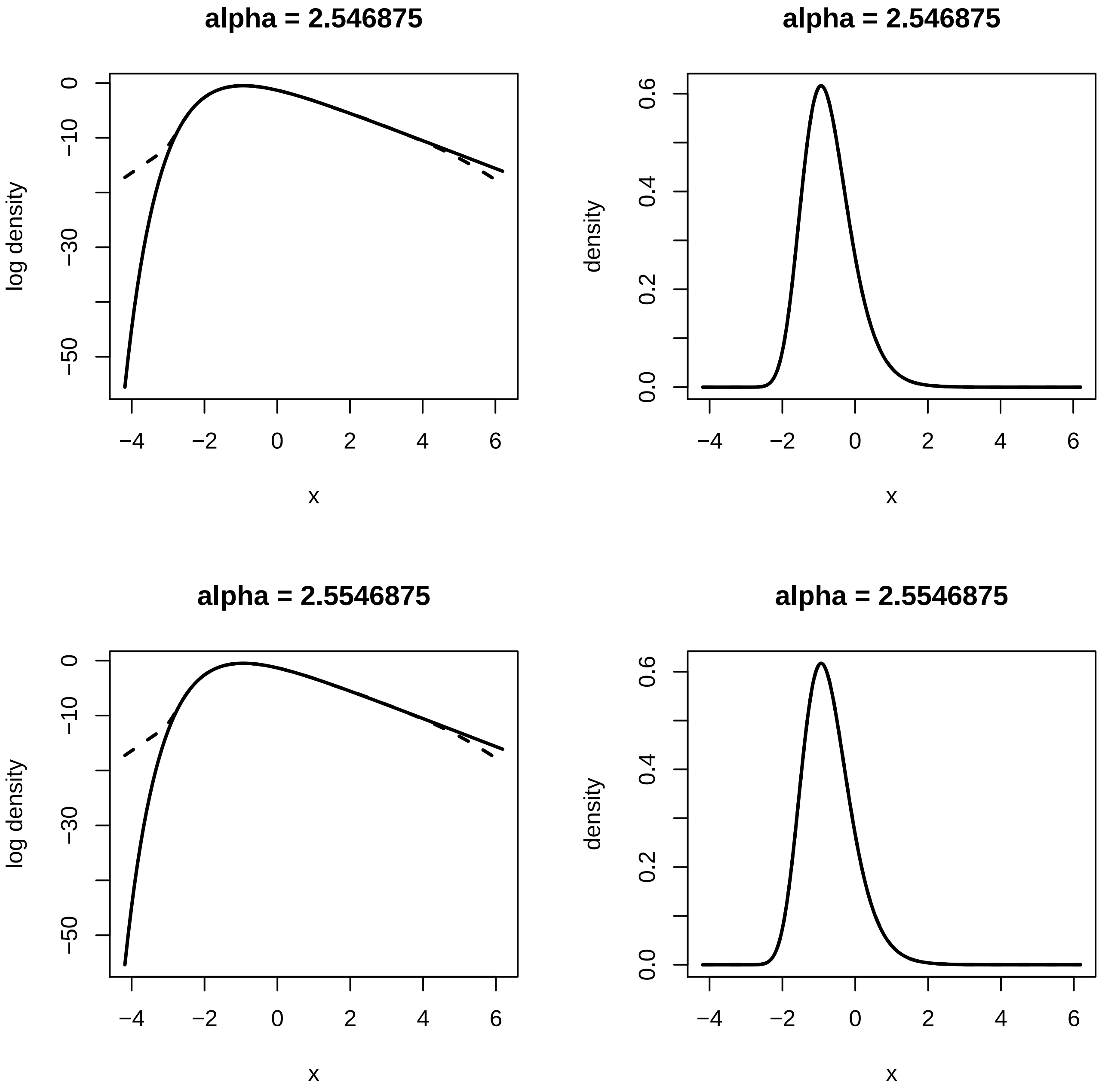


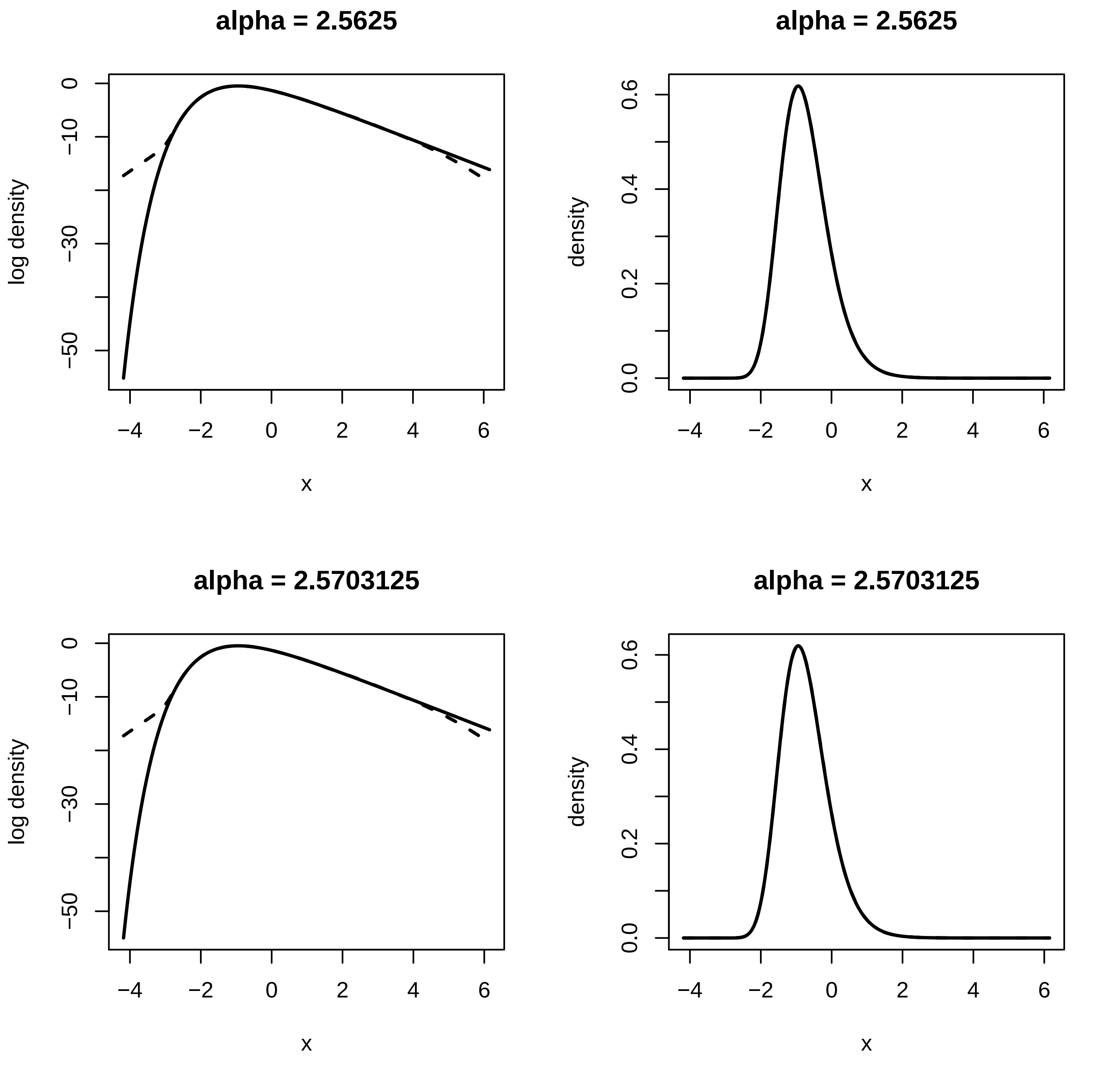
alpha = 2.484375
log density
0
−10
−30
−50
−4
−2
0
2
4
6
x
alpha = 2.484375
density
0.0
0.2
0.4
0.6
x
alpha = 2.4921875
log density
x
alpha = 2.4921875
density
x

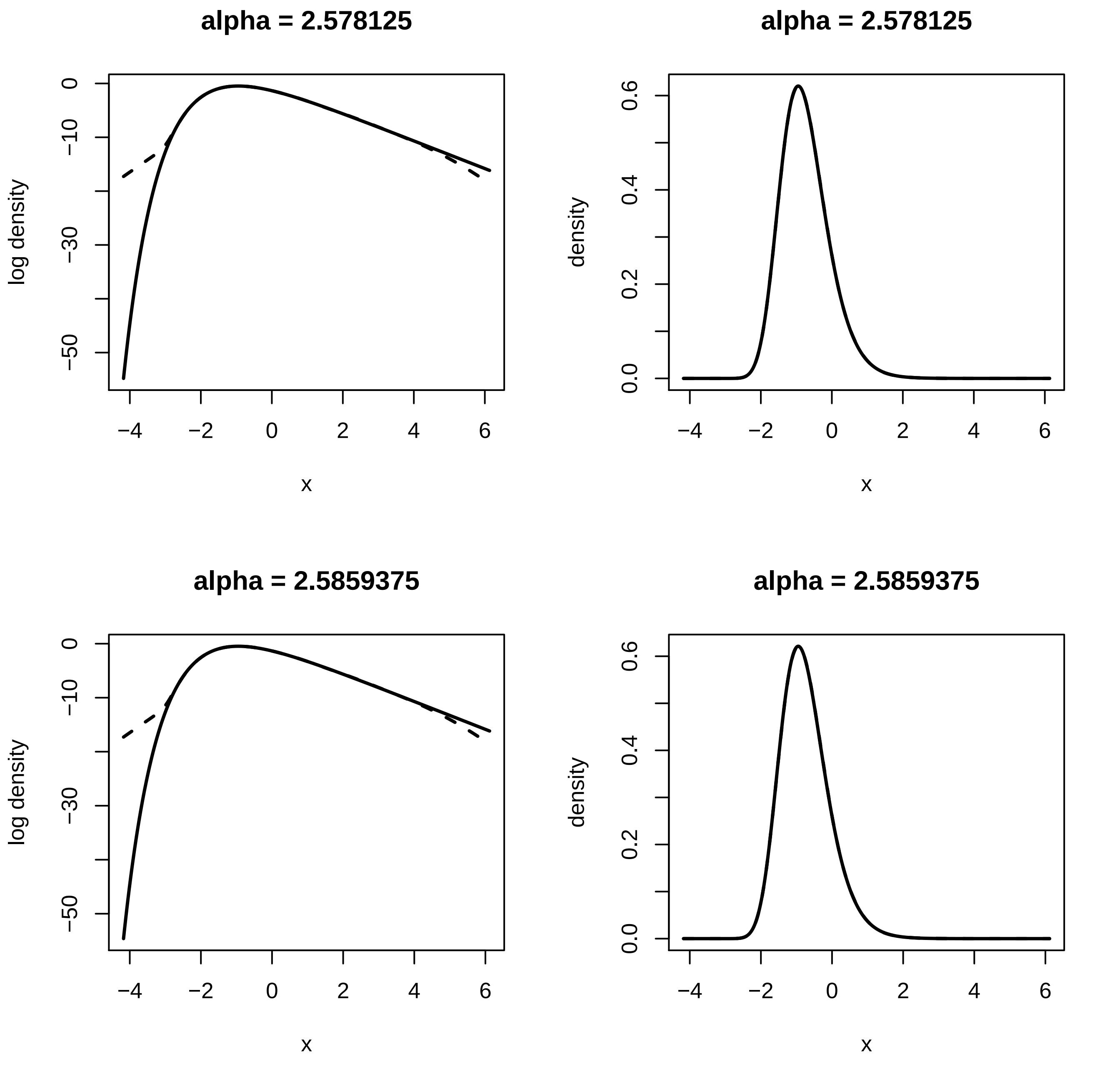

alpha = 2.5
log density
0
−10
−30
−50
−4
−2
0
2
4
6
x
alpha = 2.5
density
0.0
0.2
0.4
0.6
−4
−2
0
2
4
6
x
alpha = 2.5078125
log density
0
−10
−30
−50
−4
−2
0
2
4
6
x
alpha = 2.5078125
density
0.0
0.2
0.4
0.6
−4
−2
0
2
4
6
x

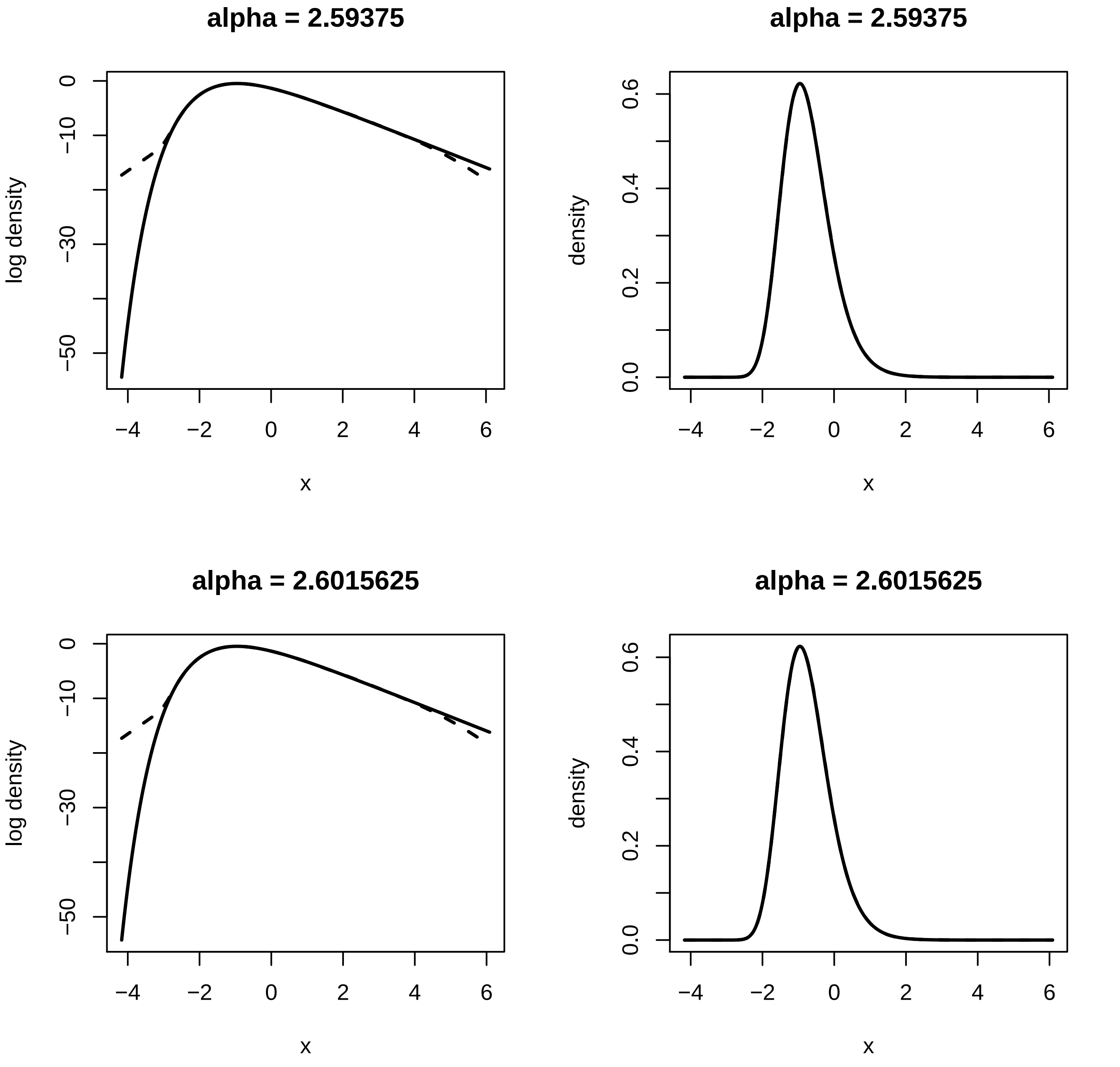

alpha = 2.515625
log density
0
−10
−30
−50
−4
−2
0
2
4
6
x
alpha = 2.515625
density
0.0
0.2
0.4
0.6
−4
−2
0
2
4
6
x
alpha = 2.5234375
log density
0
−10
−30
−50
−4
−2
0
2
4
6
x
alpha = 2.5234375
density
0.0
0.2
0.4
0.6
−4
−2
0
2
4
6
x

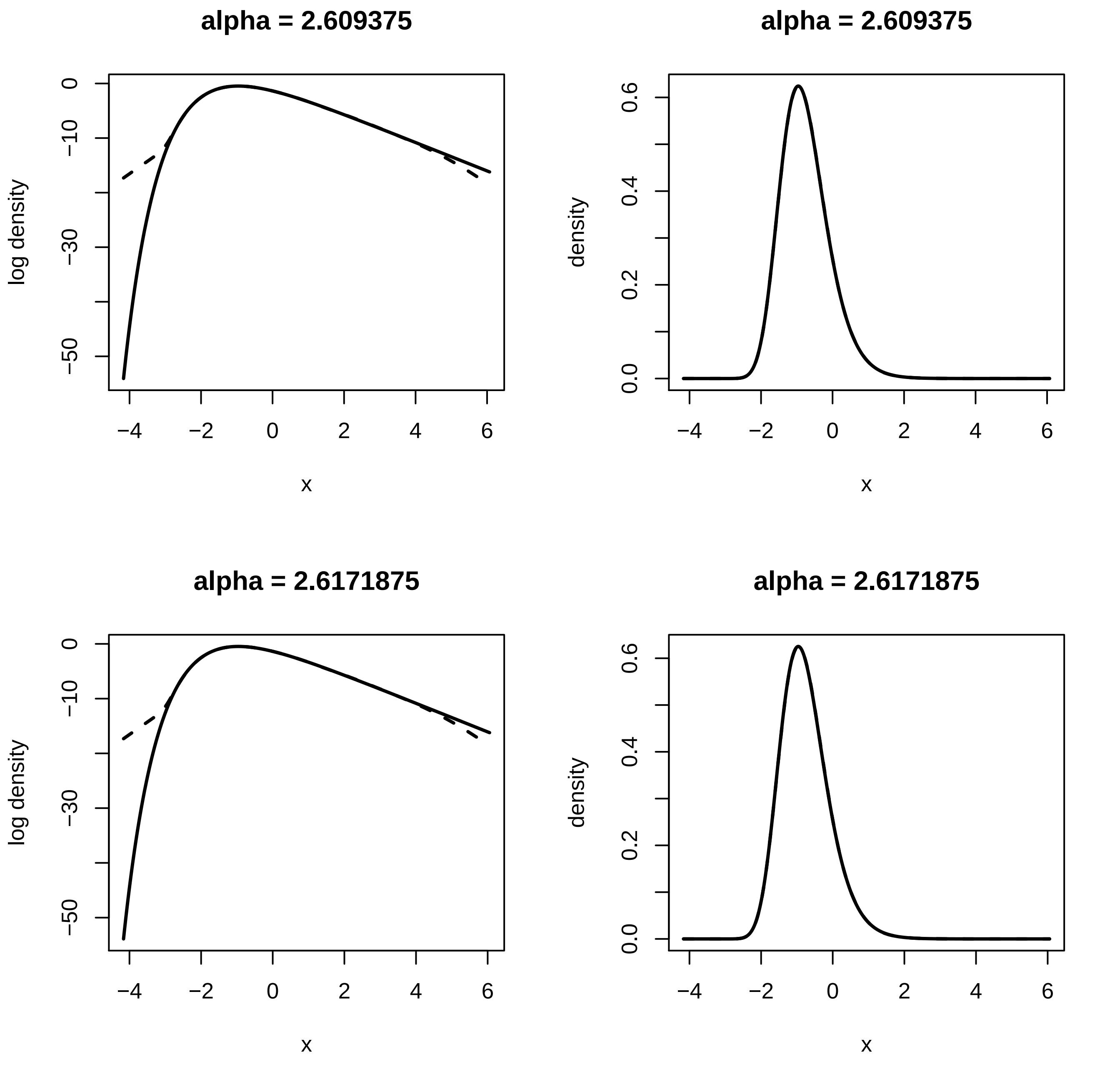

alpha = 2.53125
log density
0
−10
−30
−50
−4
−2
0
2
4
6
x
alpha = 2.53125
density
0.0
0.2
0.4
0.6
−4
−2
0
2
4
6
x
alpha = 2.5390625
log density
0
−10
−30
−50
−4
−2
0
2
4
6
x
alpha = 2.5390625
density
0.0
0.2
0.4
0.6
−4
−2
0
2
4
6
x

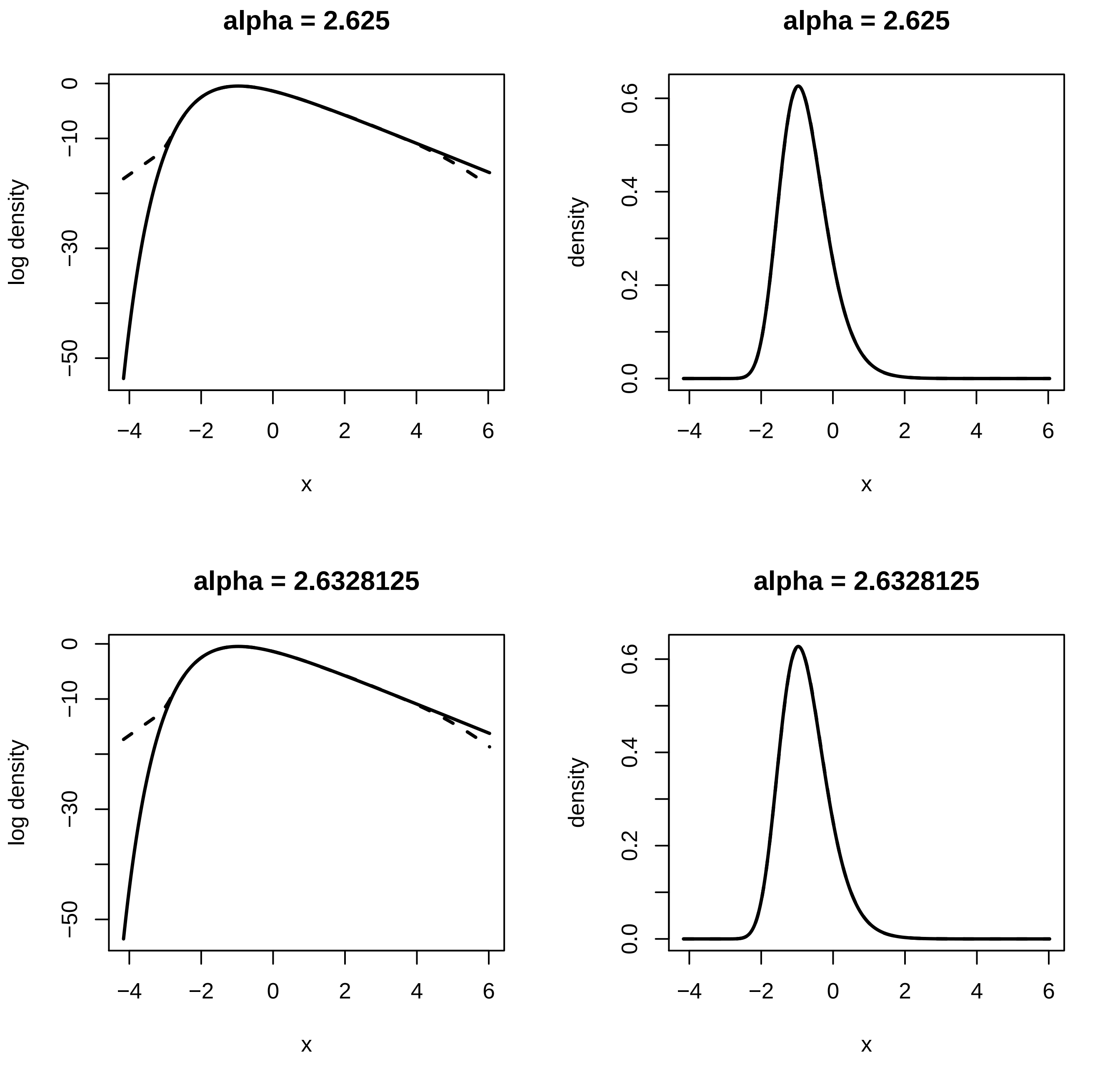

alpha = 2.546875
log density
0
−10
−30
−50
−4
−2
0
2
4
6
x
alpha = 2.546875
density
0.0
0.2
0.4
0.6
x
alpha = 2.5546875
log density
x
alpha = 2.5546875
density
x

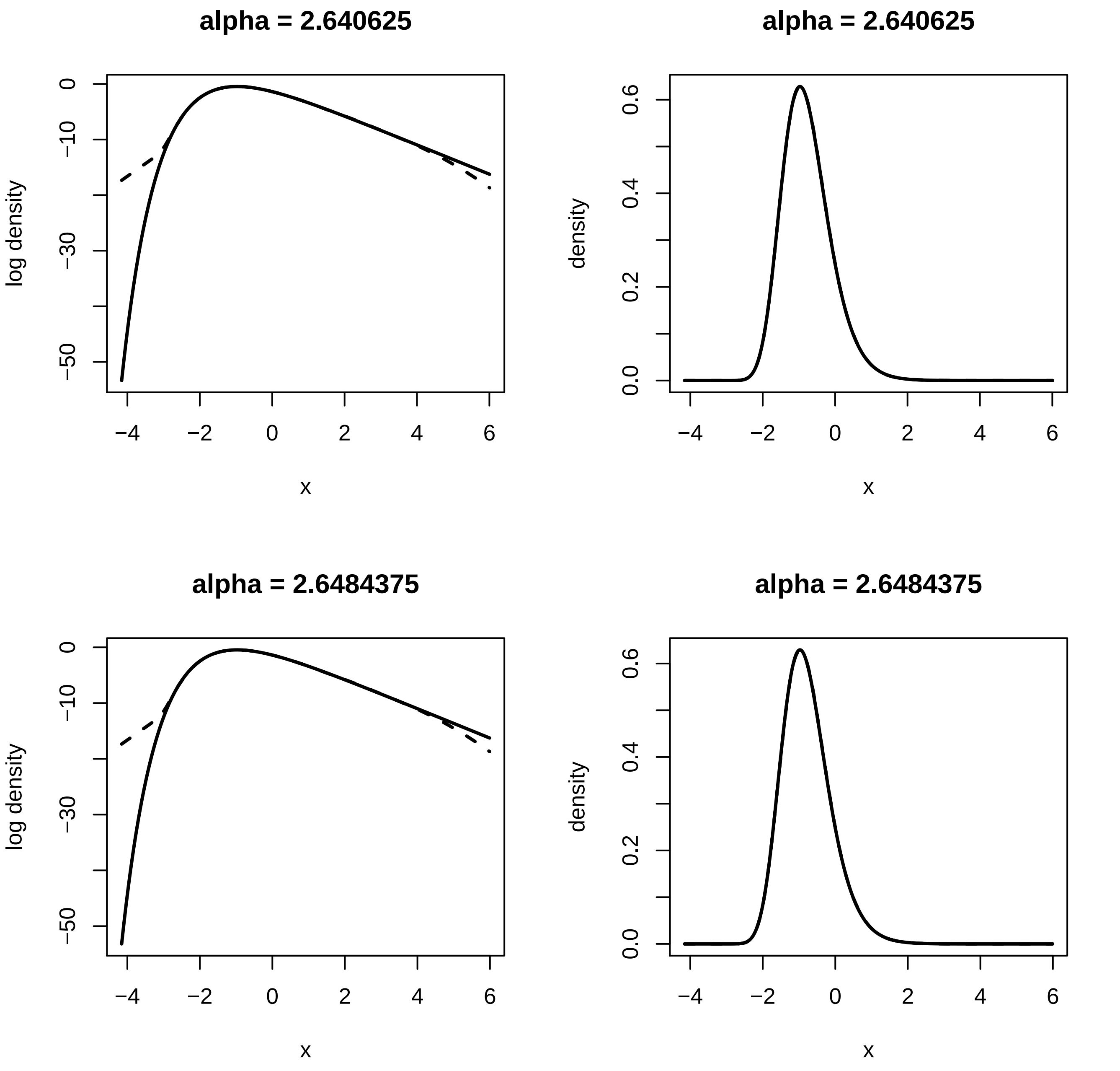

alpha = 2.5625
log density
0
−10
−30
−50
−4
−2
0
2
4
6
x
alpha = 2.5625
density
0.0
0.2
0.4
0.6
−4
−2
0
2
4
6
x
alpha = 2.5703125
log density
0
−10
−30
−50
−4
−2
0
2
4
6
x
alpha = 2.5703125
density
0.0
0.2
0.4
0.6
−4
−2
0
2
4
6
x

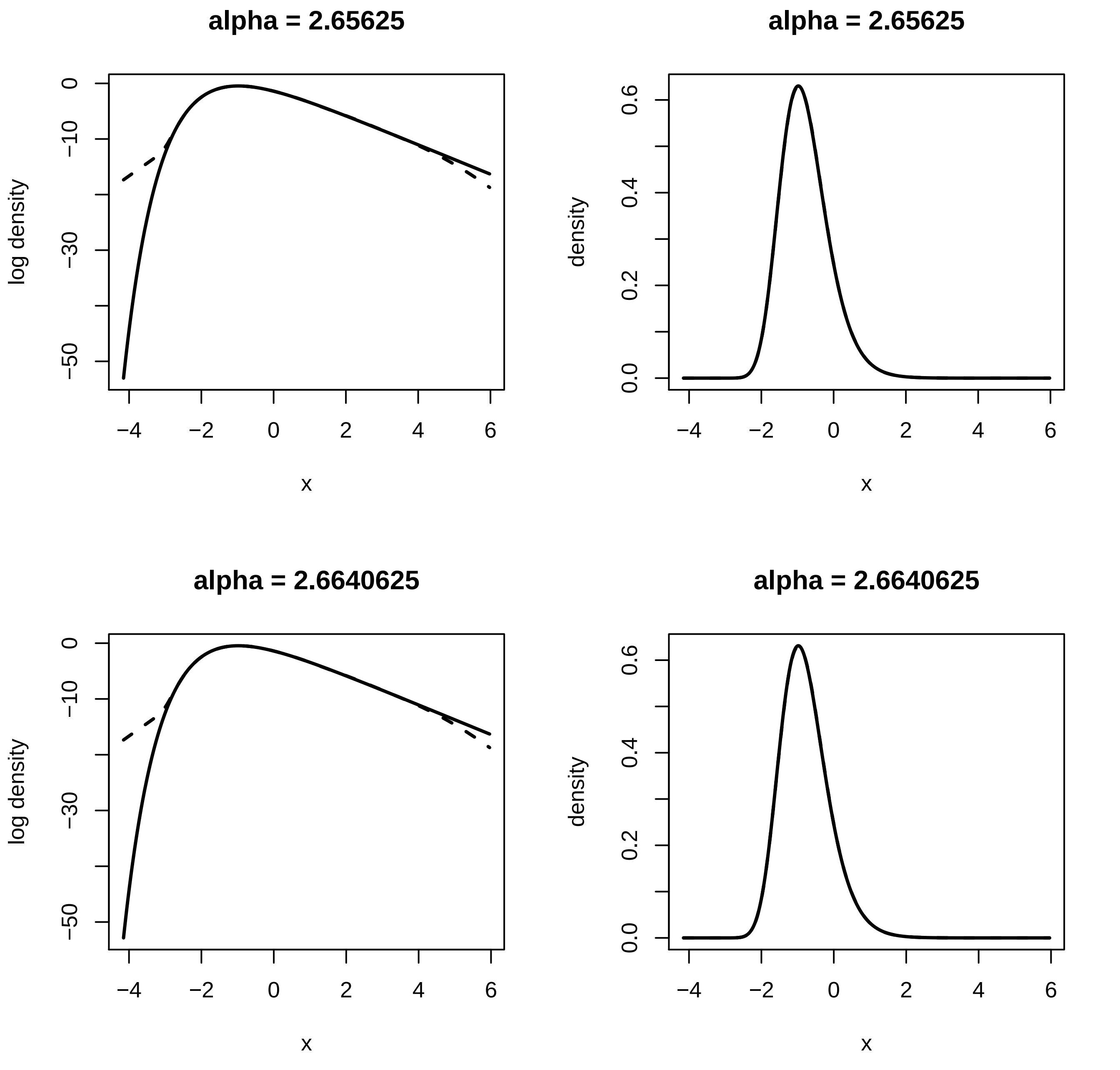

alpha = 2.578125
log density
0
−10
−30
−50
−4
−2
0
2
4
6
x
alpha = 2.578125
density
0.0
0.2
0.4
0.6
x
alpha = 2.5859375
log density
x
alpha = 2.5859375
density
x

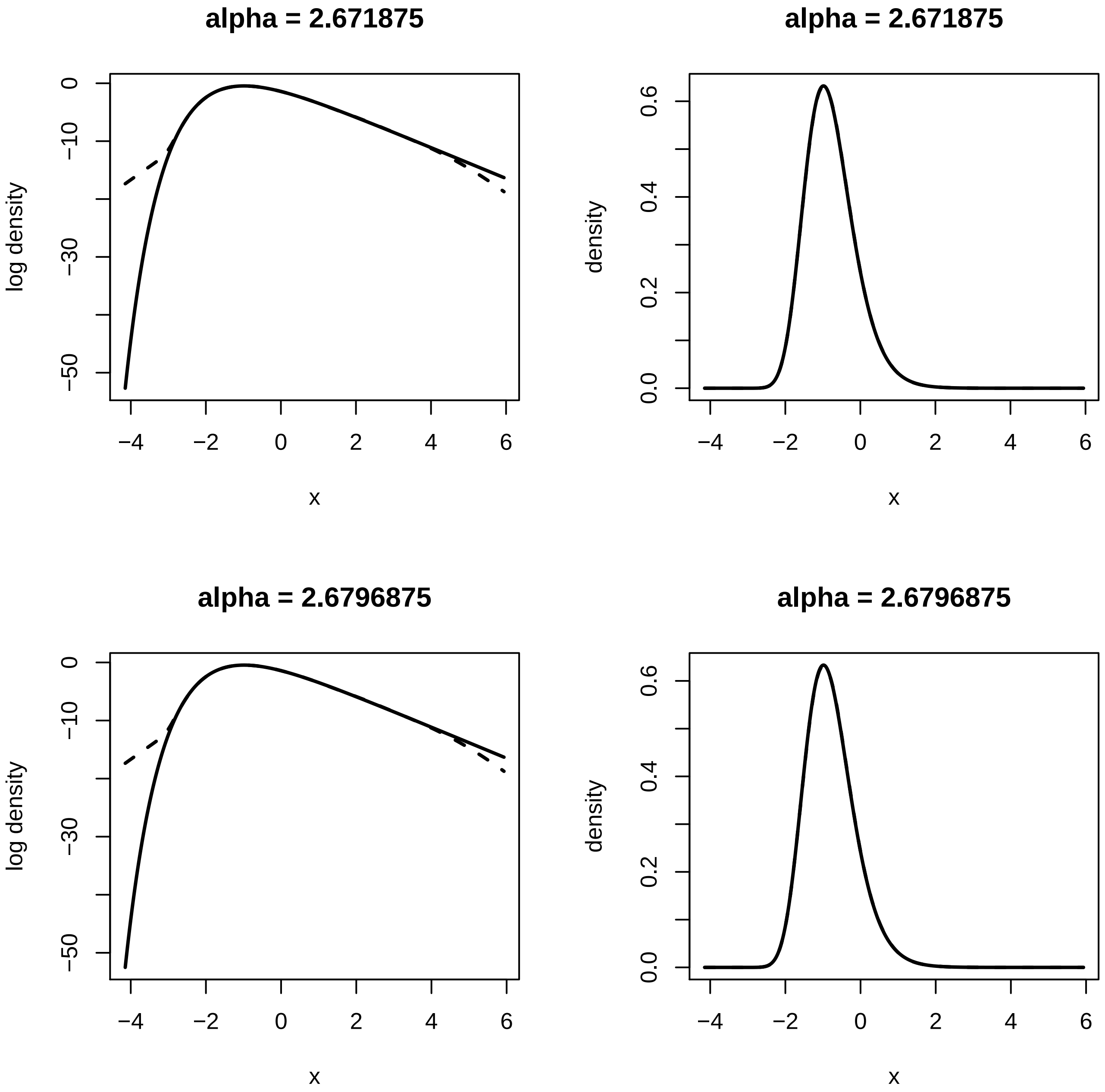

alpha = 2.59375
log density
0
−10
−30
−50
−4
−2
0
2
4
6
x
alpha = 2.59375
density
0.0
0.2
0.4
0.6
x
alpha = 2.6015625
log density
x
alpha = 2.6015625
density
x

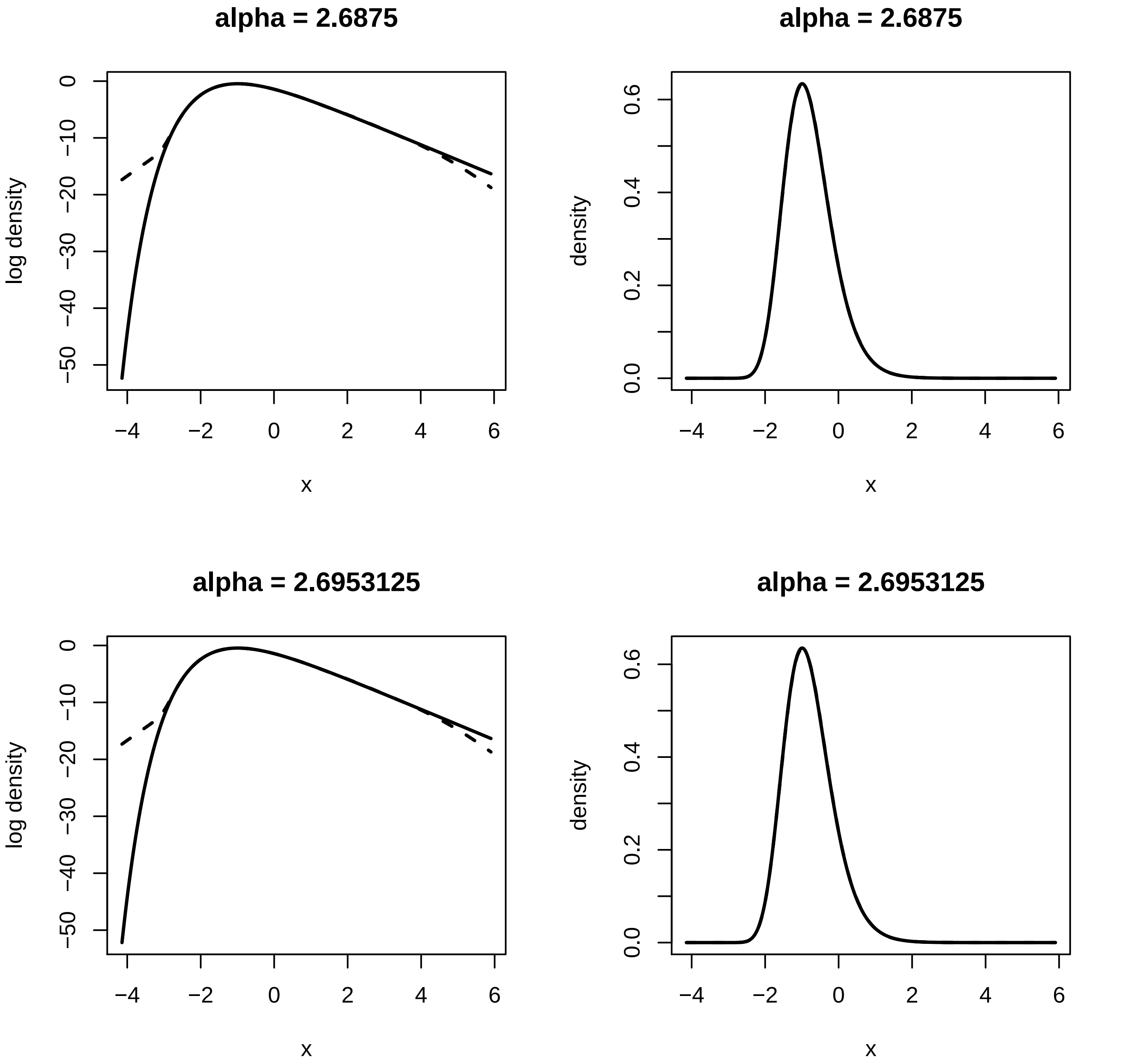

alpha = 2.609375
log density
0
−10
−30
−50
−4
−2
0
2
4
6
x
alpha = 2.609375
density
0.0
0.2
0.4
0.6
−4
−2
0
2
4
6
x
alpha = 2.6171875
log density
0
−10
−30
−50
−4
−2
0
2
4
6
x
alpha = 2.6171875
density
0.0
0.2
0.4
0.6
−4
−2
0
2
4
6
x

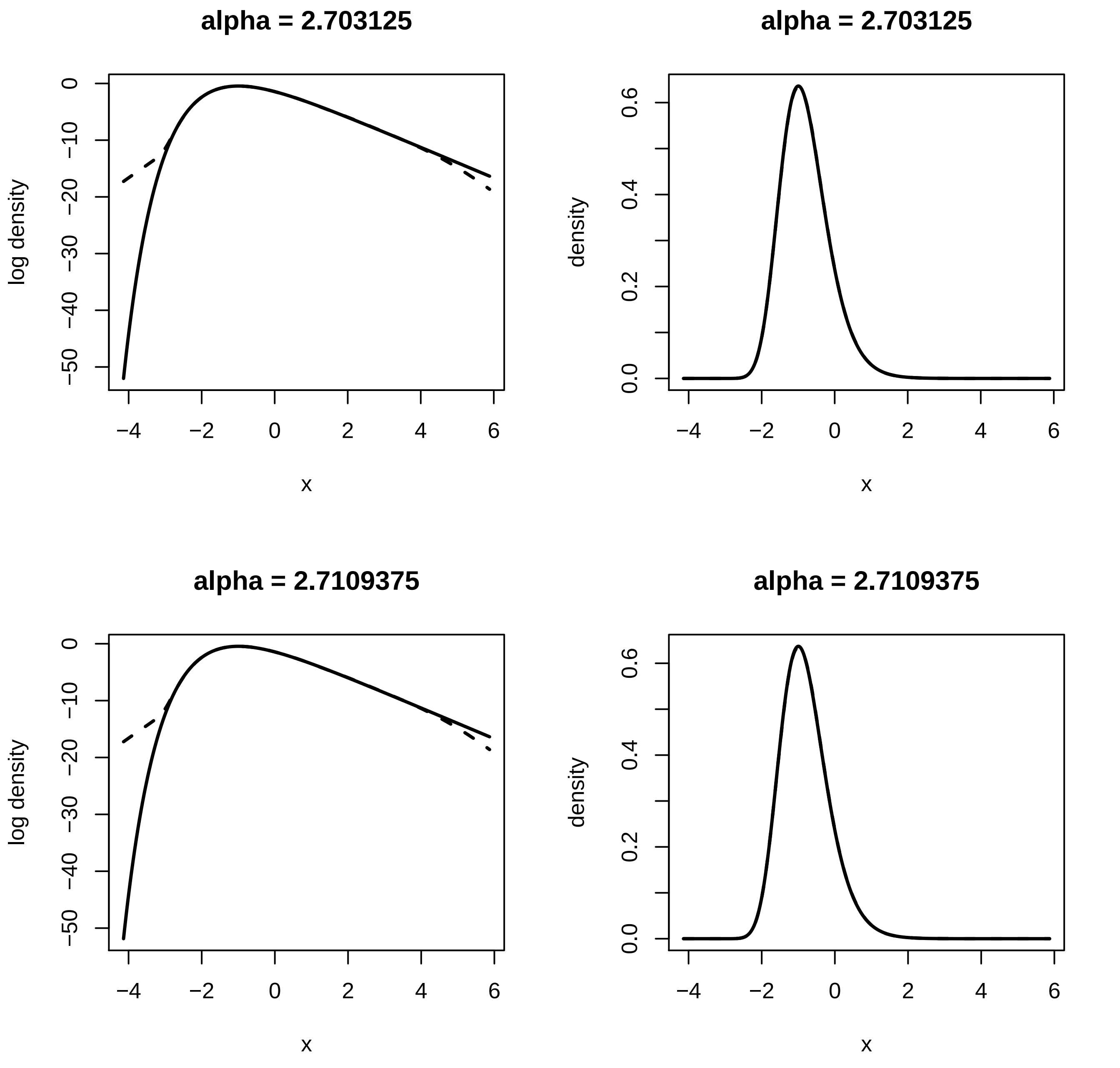

alpha = 2.625
log density
0
−10
−30
−50
−4
−2
0
2
4
6
x
alpha = 2.625
density
0.0
0.2
0.4
0.6
−4
−2
0
2
4
6
x
alpha = 2.6328125
log density
0
−10
−30
−50
−4
−2
0
2
4
6
x
alpha = 2.6328125
density
0.0
0.2
0.4
0.6
−4
−2
0
2
4
6
x


alpha = 2.640625
log density
0
−10
−30
−50
−4
−2
0
2
4
6
x
alpha = 2.640625
density
0.0
0.2
0.4
0.6
x
alpha = 2.6484375
log density
x
alpha = 2.6484375
density
x

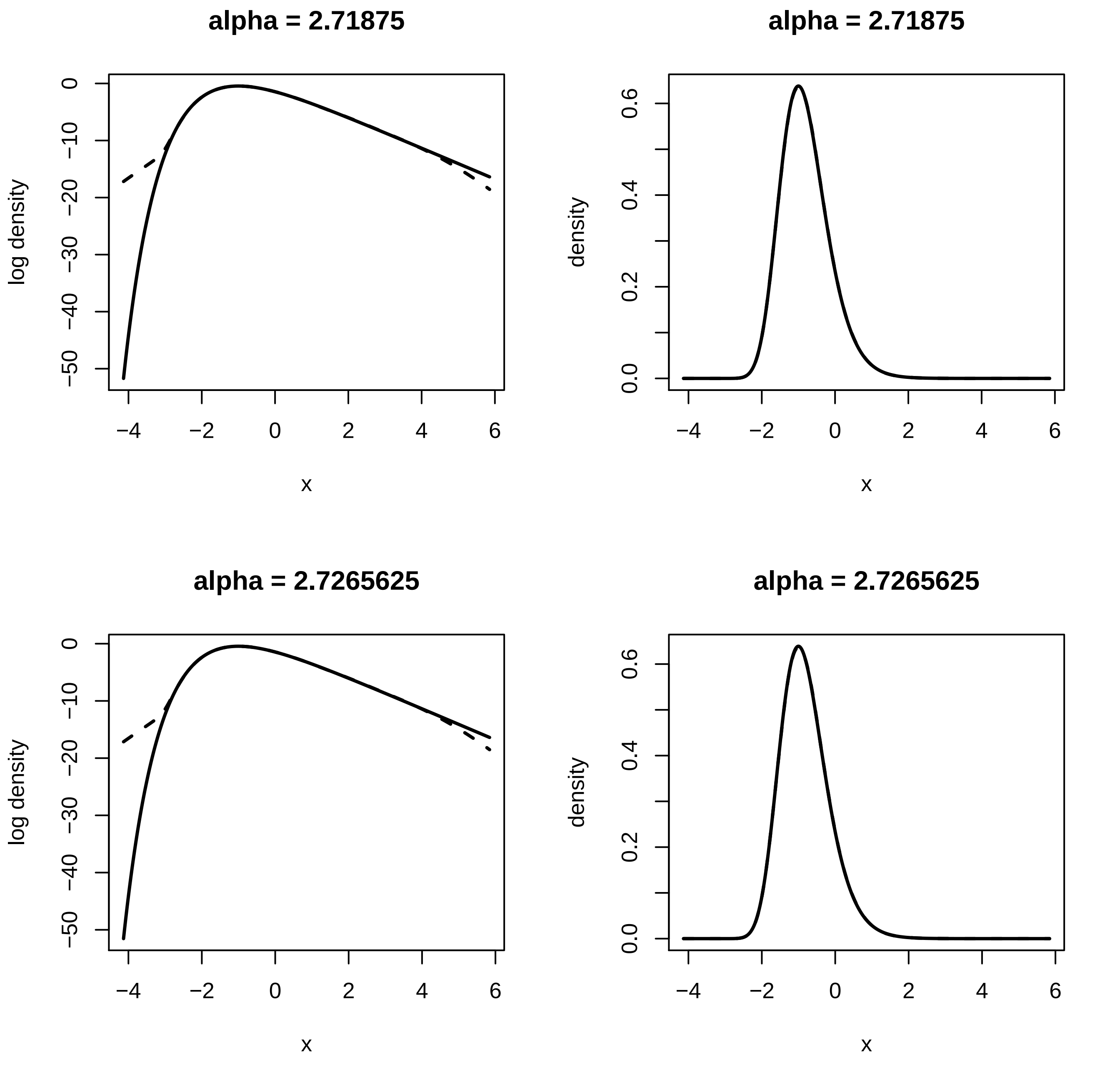

alpha = 2.65625
log density
0
−10
−30
−50
−4
−2
0
2
4
6
x
alpha = 2.65625
density
0.0
0.2
0.4
0.6
−4
−2
0
2
4
6
x
alpha = 2.6640625
log density
0
−10
−30
−50
−4
−2
0
2
4
6
x
alpha = 2.6640625
density
0.0
0.2
0.4
0.6
−4
−2
0
2
4
6
x

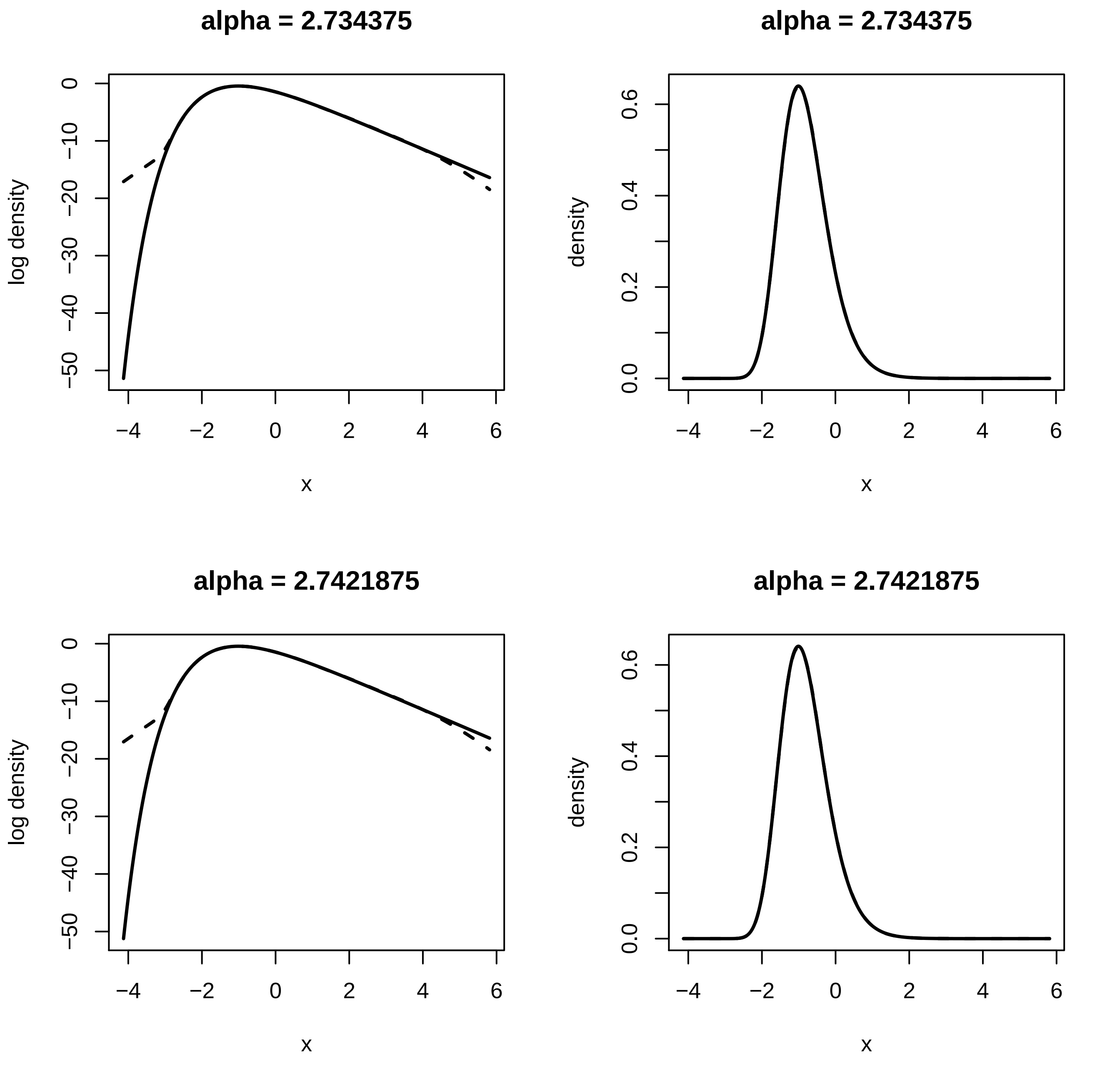

alpha = 2.671875
log density
0
−10
−30
−50
−4
−2
0
2
4
6
x
alpha = 2.671875
density
0.0
0.2
0.4
0.6
−4
−2
0
2
4
6
x
alpha = 2.6796875
log density
0
−10
−30
−50
−4
−2
0
2
4
6
x
alpha = 2.6796875
density
0.0
0.2
0.4
0.6
−4
−2
0
2
4
6
x

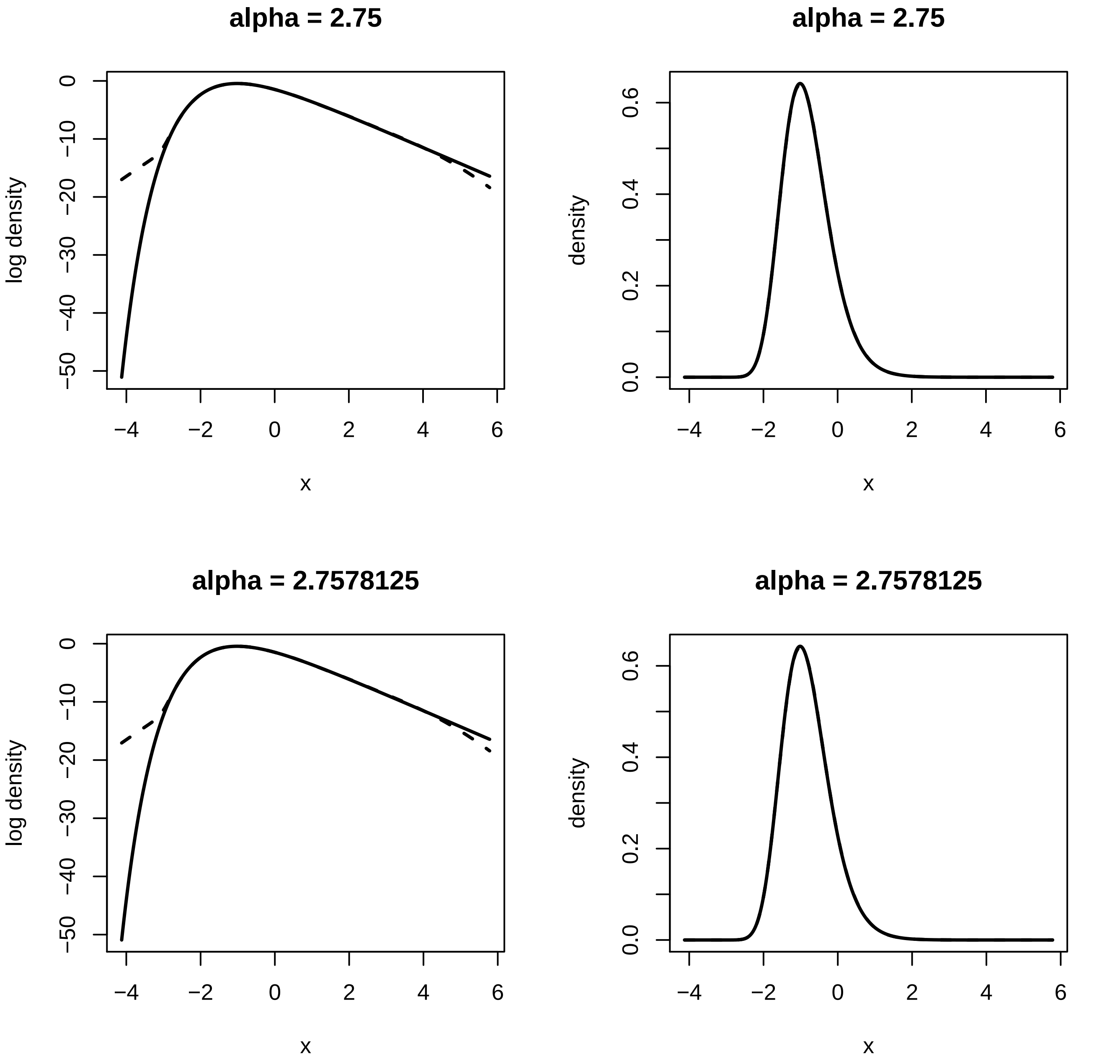

alpha = 2.6875
log density
0
−10
−20
−30
−40
−50
−4
−2
0
2
4
6
x
alpha = 2.6875
density
0.0
0.2
0.4
0.6
−4
−2
0
2
4
6
x
alpha = 2.6953125
log density
0
−10
−20
−30
−40
−50
−4
−2
0
2
4
6
x
alpha = 2.6953125
density
0.0
0.2
0.4
0.6
−4
−2
0
2
4
6
x



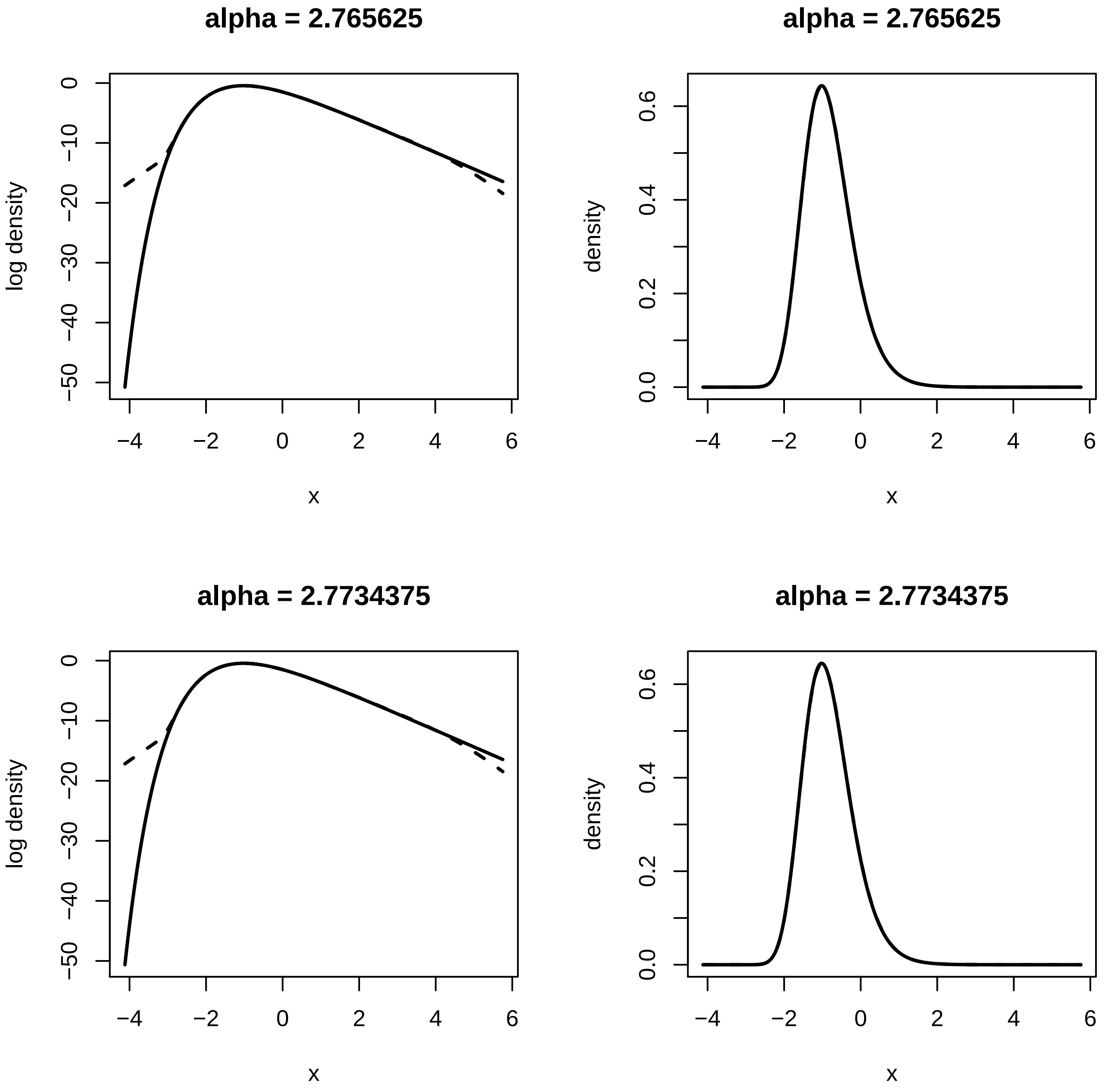
alpha = 2.703125
log density
x
alpha = 2.703125
density
x
alpha = 2.7109375
log density
x
alpha = 2.7109375
density
x


alpha = 2.71875
alpha = 2.71875
alpha = 2.7265625
alpha = 2.7265625
log density
density
x
0
−10
−20
−30
−40
−50
0.0
0.2
0.4
0.6
−4
−2
0
2
4
6

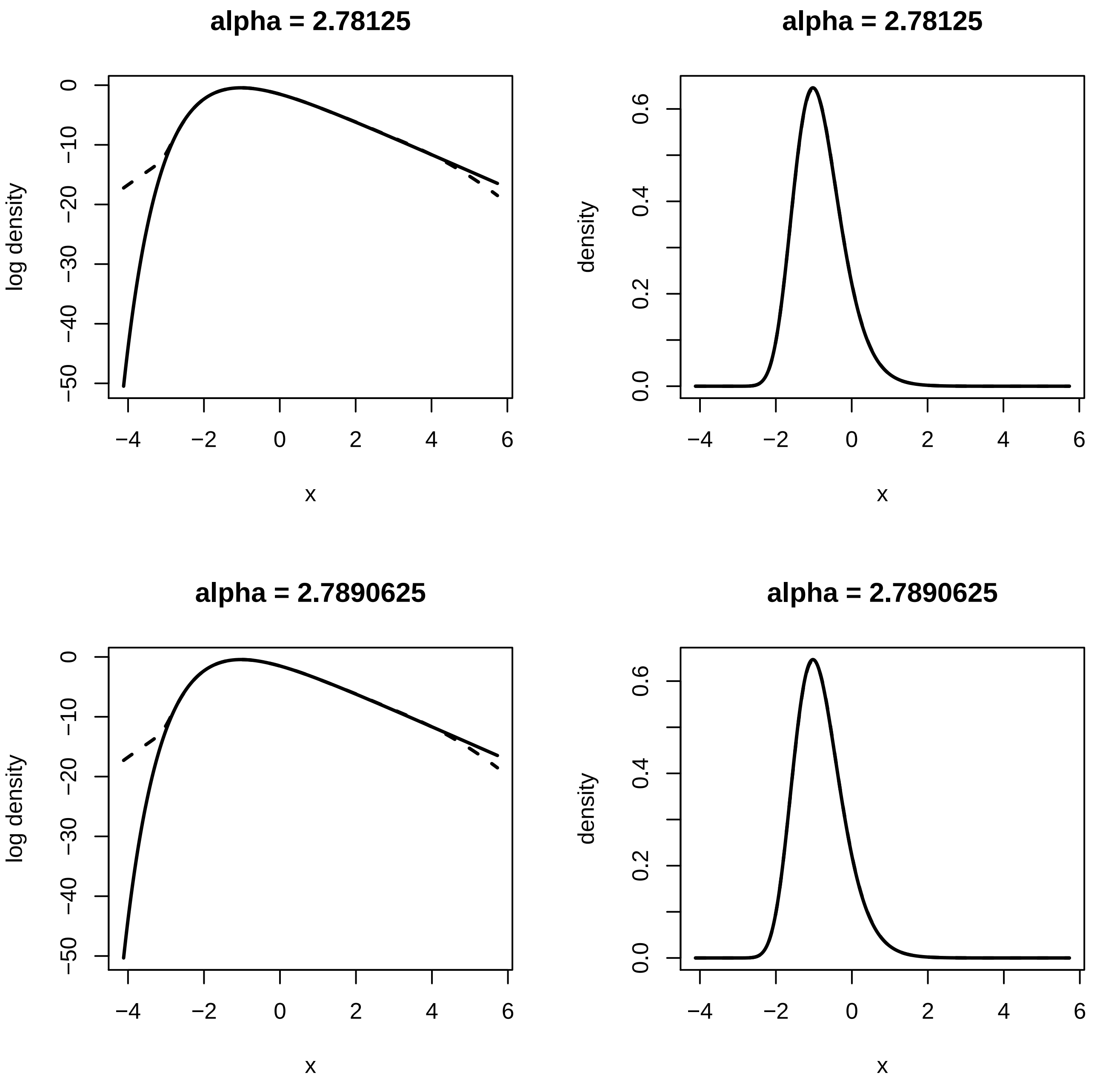

alpha = 2.734375
log density
x
alpha = 2.734375
density
x
alpha = 2.7421875
log density
x
alpha = 2.7421875
density
x

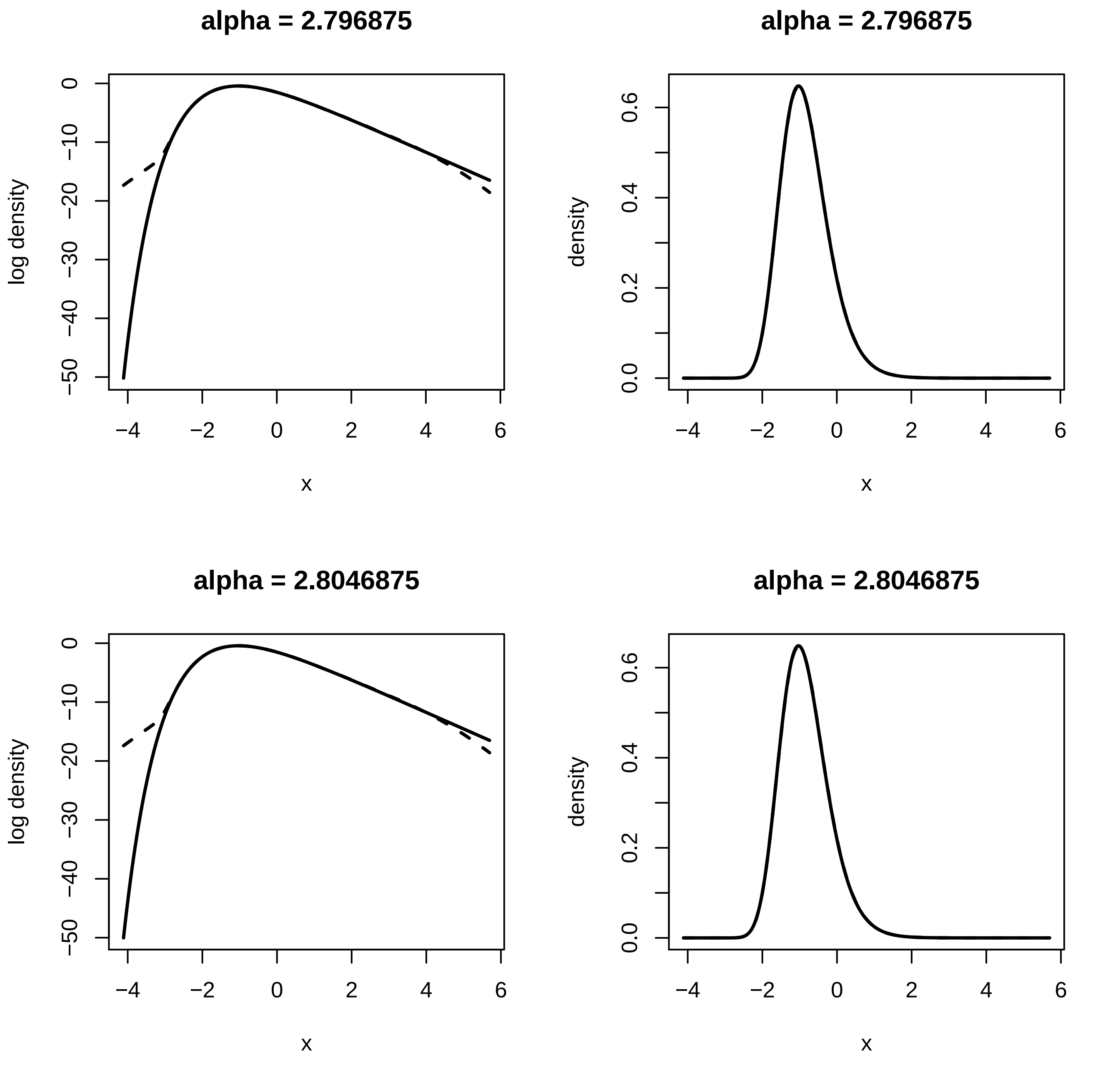


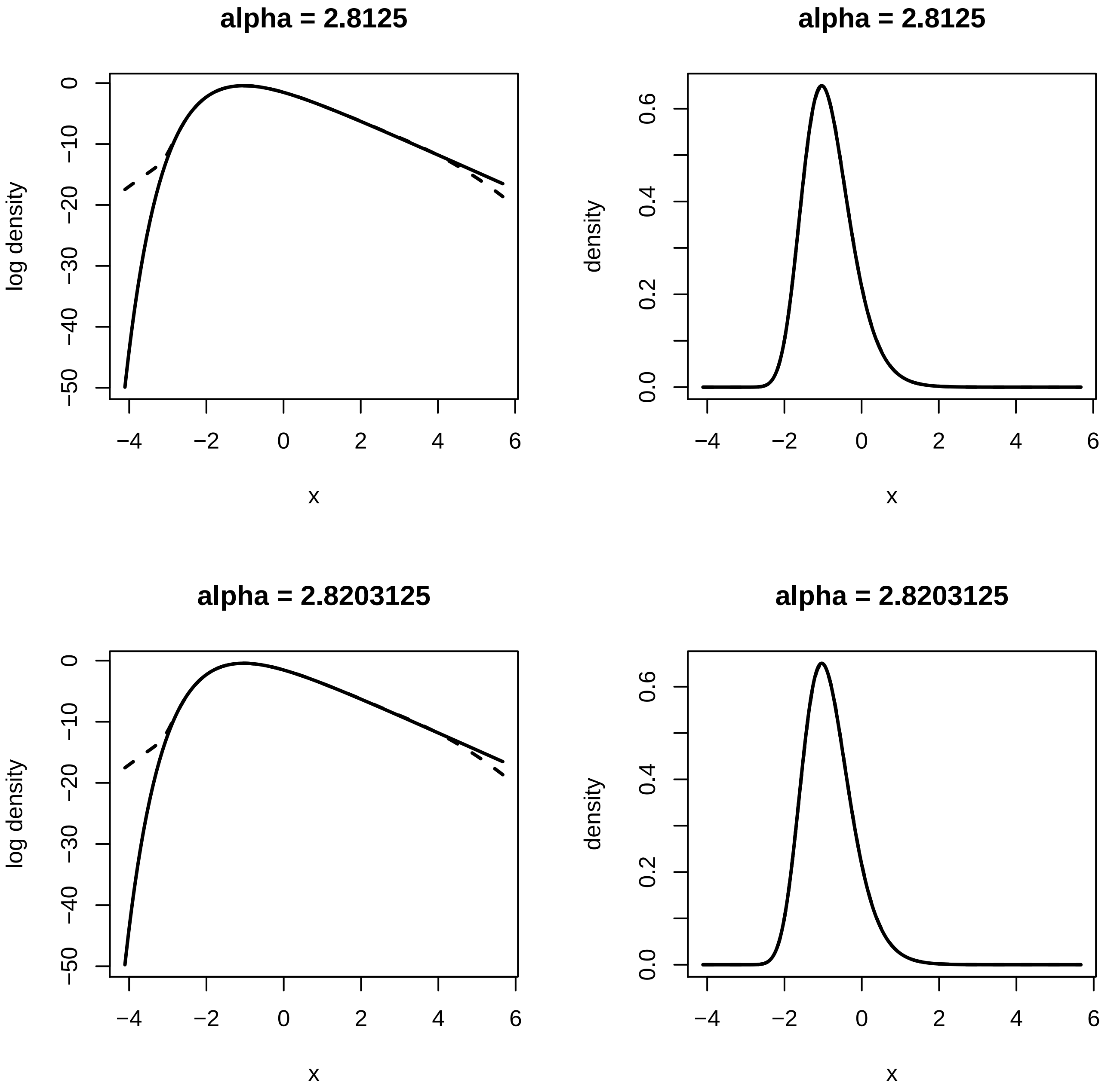
alpha = 2.75
log density
0
−10
−20
−30
−40
−50
−4
−2
0
2
4
6
x
alpha = 2.75
density
0.0
0.2
0.4
0.6
−4
−2
0
2
4
6
x
alpha = 2.7578125
log density
0
−10
−20
−30
−40
−50
−4
−2
0
2
4
6
x
alpha = 2.7578125
density
0.0
0.2
0.4
0.6
−4
−2
0
2
4
6
x

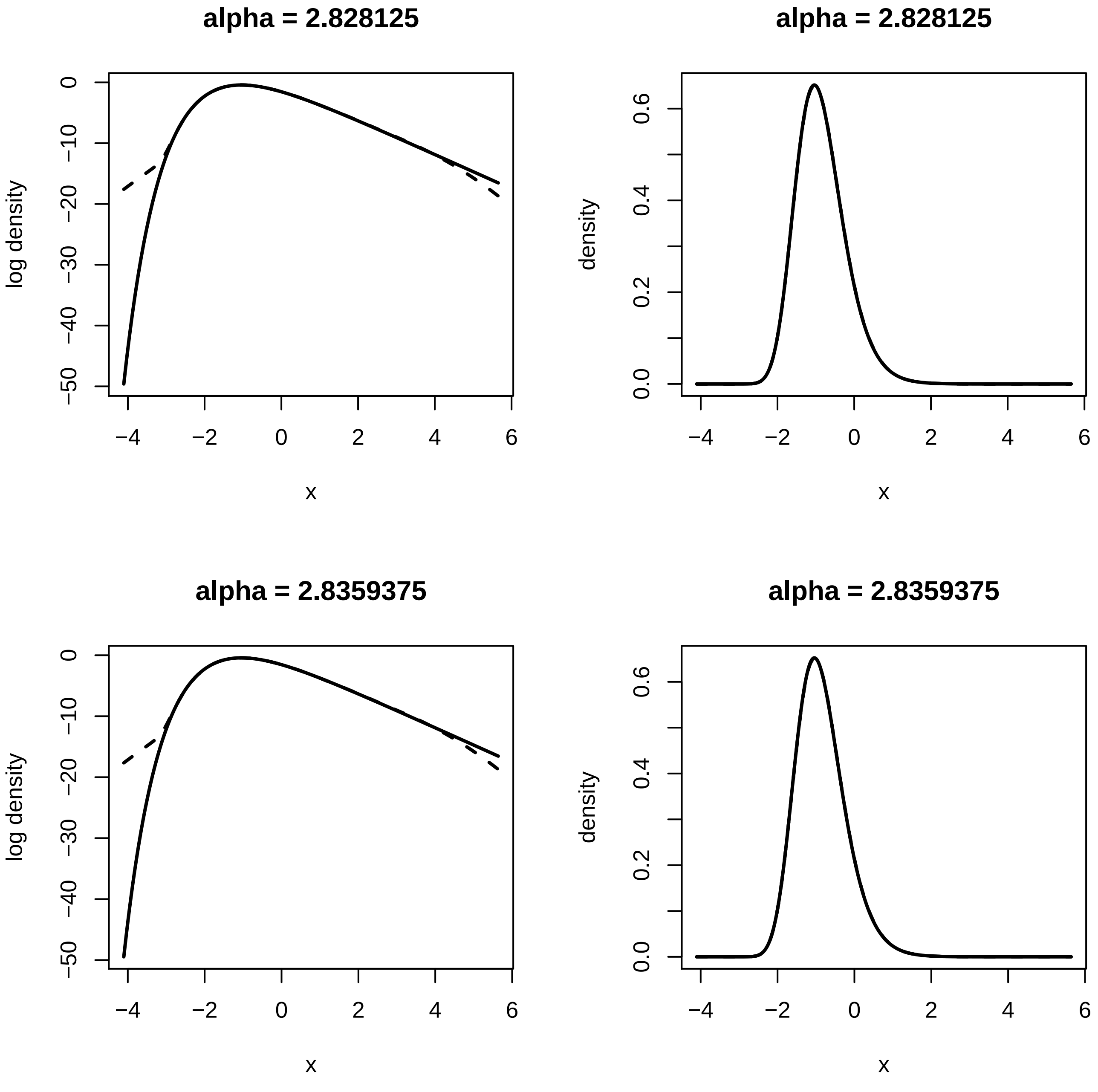


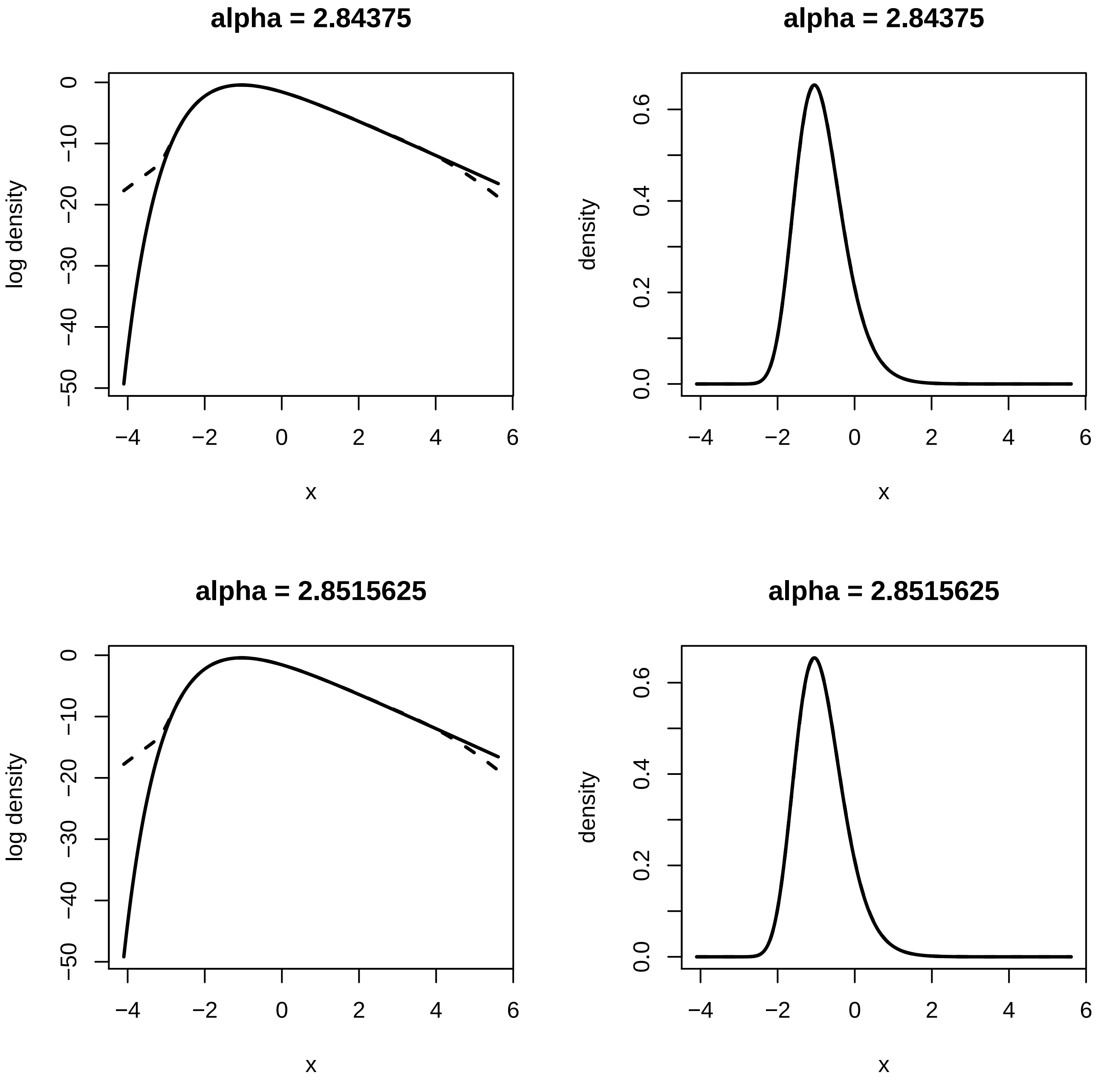
alpha = 2.765625
log density
0
−10
−20
−30
−40
−50
−4
−2
0
2
4
6
x
alpha = 2.765625
density
0.0
0.2
0.4
0.6
−4
−2
0
2
4
6
x
alpha = 2.7734375
log density
0
−10
−20
−30
−40
−50
−4
−2
0
2
4
6
x
alpha = 2.7734375
density
0.0
0.2
0.4
0.6
−4
−2
0
2
4
6
x

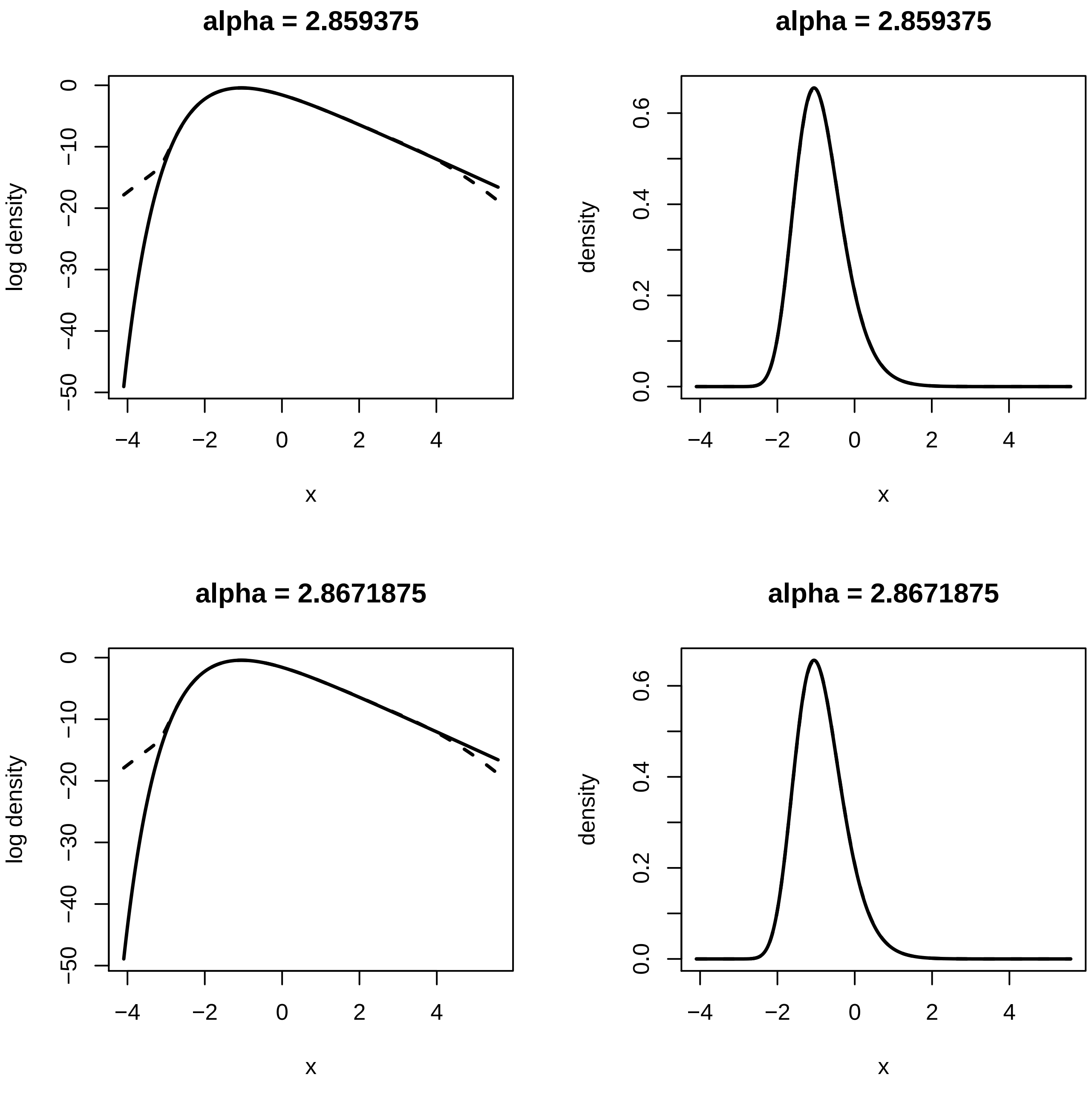

alpha = 2.78125
log density
0
−10
−20
−30
−40
−50
−4
−2
0
2
4
6
x
alpha = 2.78125
density
0.0
0.2
0.4
0.6
−4
−2
0
2
4
6
x
alpha = 2.7890625
log density
0
−10
−20
−30
−40
−50
−4
−2
0
2
4
6
x
alpha = 2.7890625
density
0.0
0.2
0.4
0.6
−4
−2
0
2
4
6
x

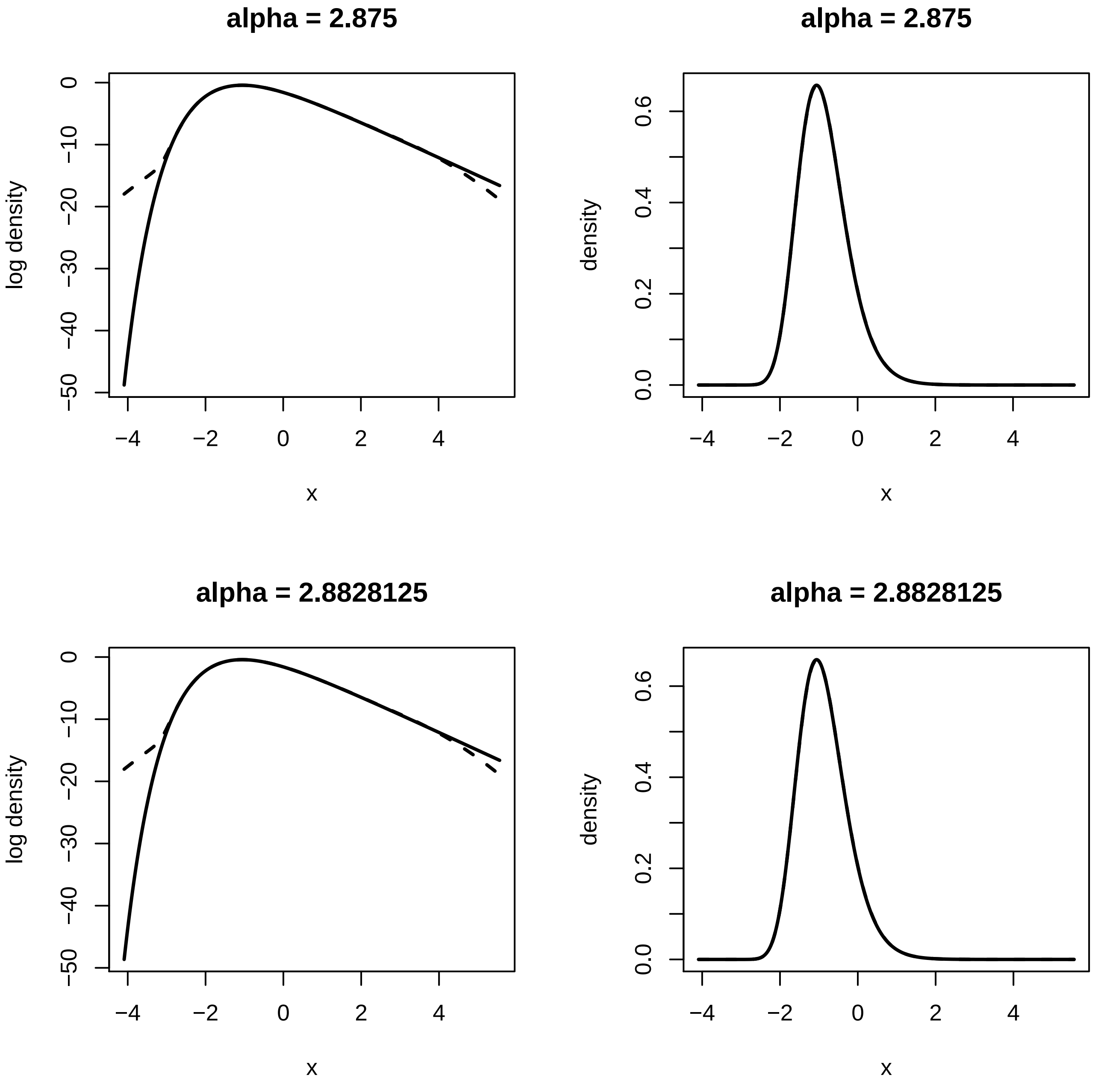

alpha = 2.796875
log density
x
alpha = 2.796875
density
x
alpha = 2.8046875
log density
x
alpha = 2.8046875
density
x

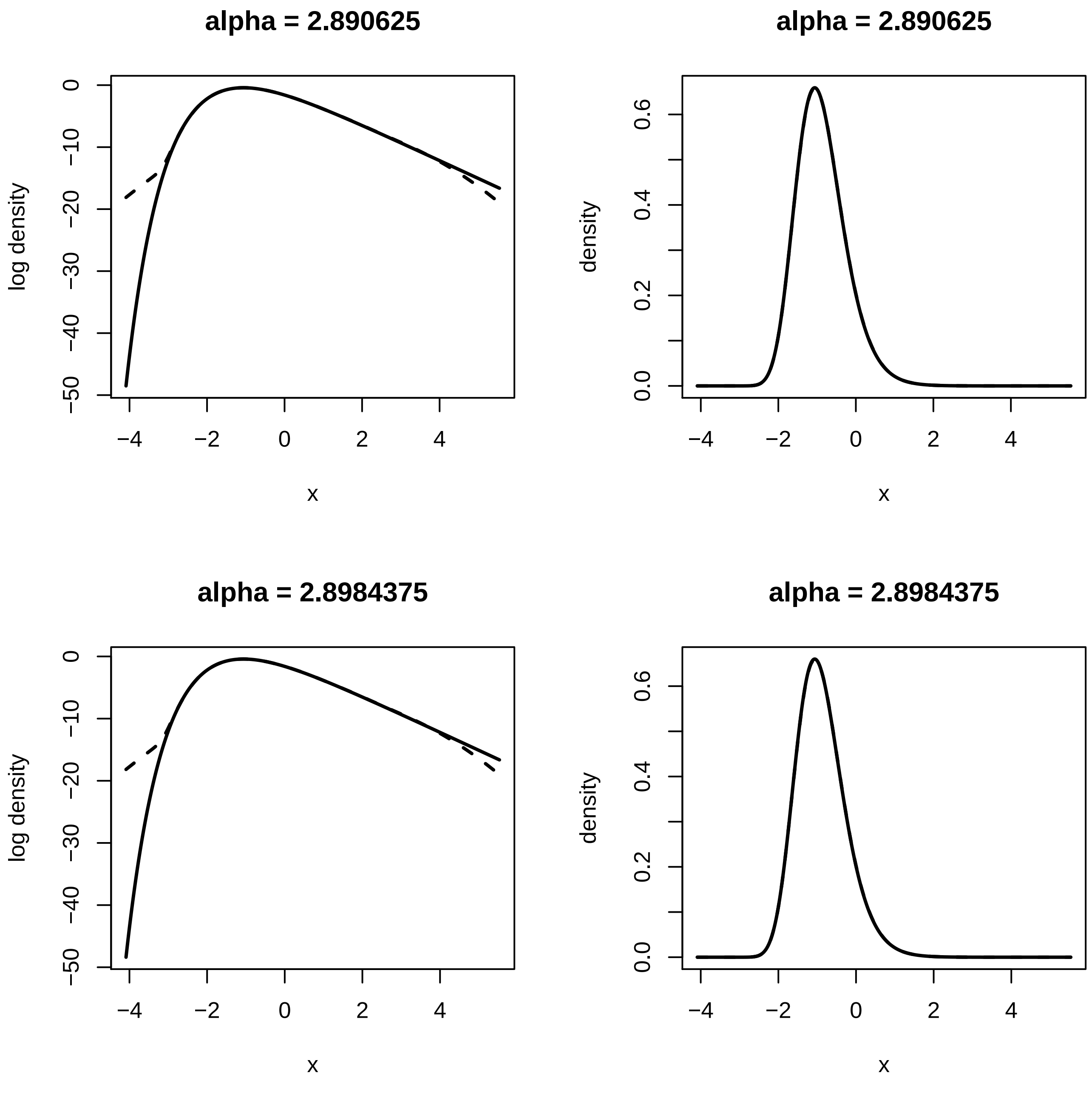

alpha = 2.8125
log density
0
−10
−20
−30
−40
−50
−4
−2
0
2
4
6
x
alpha = 2.8125
density
0.0
0.2
0.4
0.6
−4
−2
0
2
4
6
x
alpha = 2.8203125
log density
0
−10
−20
−30
−40
−50
−4
−2
0
2
4
6
x
alpha = 2.8203125
density
0.0
0.2
0.4
0.6
−4
−2
0
2
4
6
x

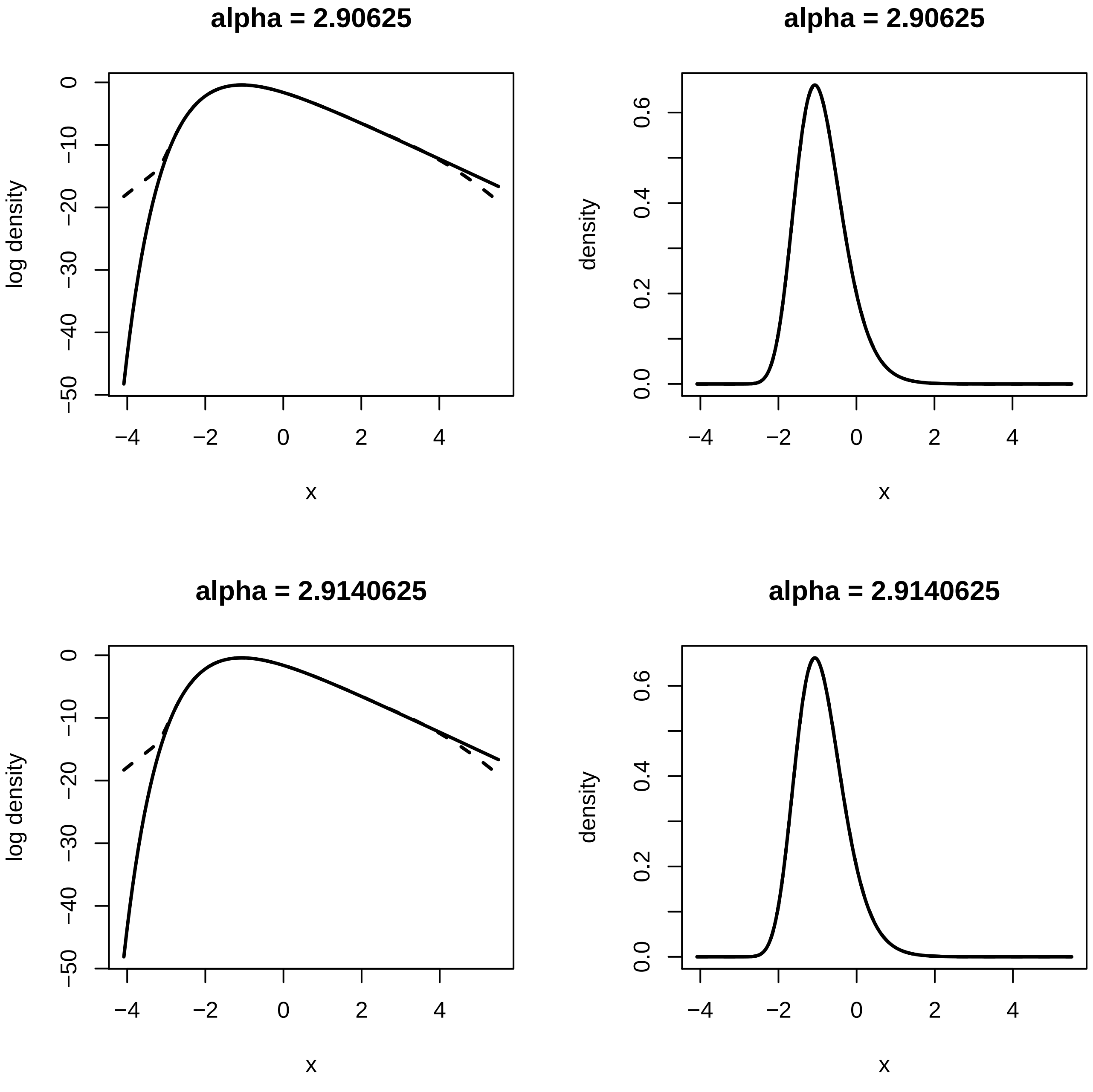

alpha = 2.828125
log density
x
alpha = 2.828125
density
x
alpha = 2.8359375
log density
x
alpha = 2.8359375
density
x

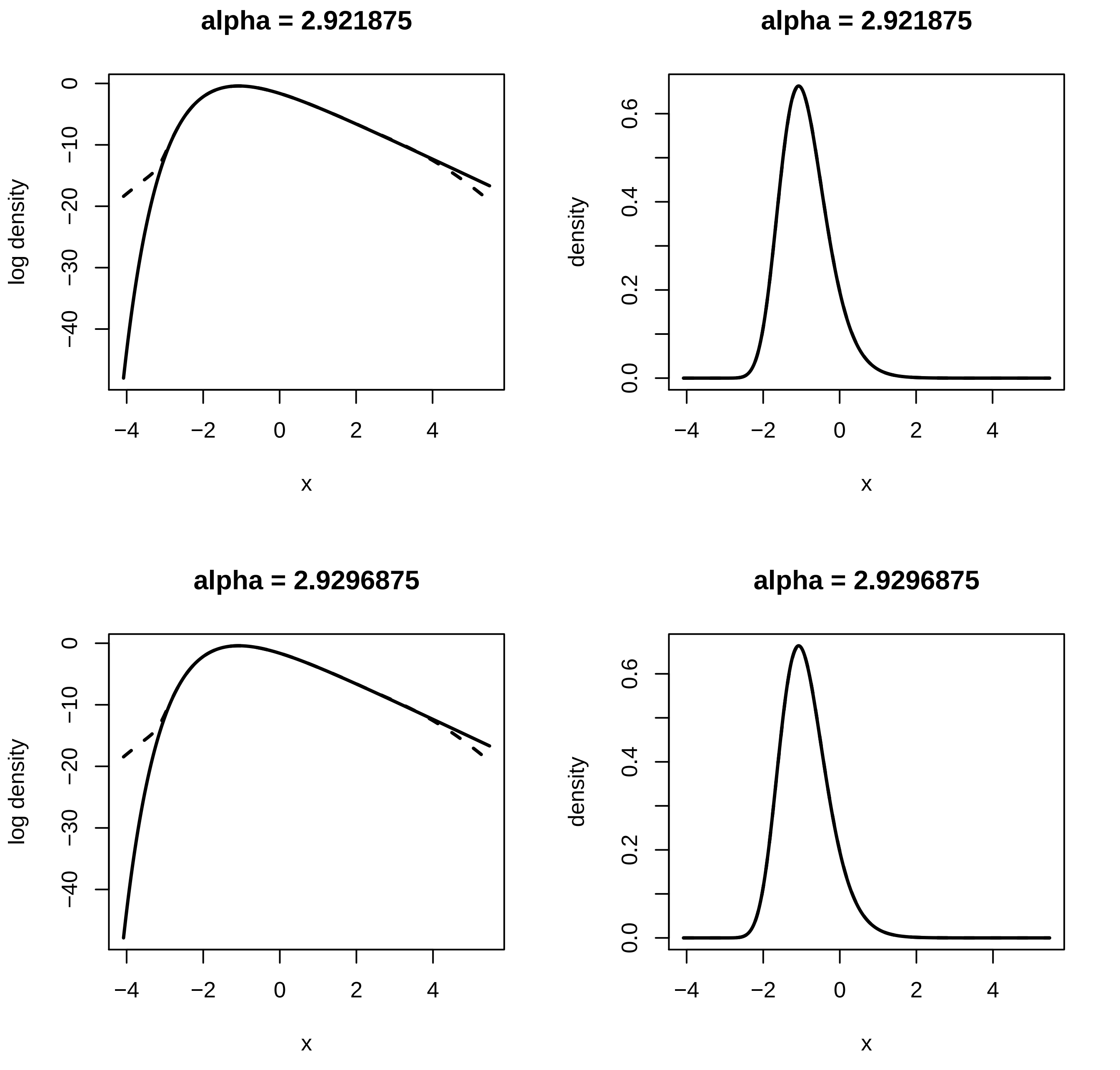

alpha = 2.84375
alpha = 2.84375
log density
density
x
x
alpha = 2.8515625
alpha = 2.8515625
log density
density
x
x


alpha = 2.859375
log density
x
alpha = 2.859375
density
x
alpha = 2.8671875
log density
x
alpha = 2.8671875
density
x

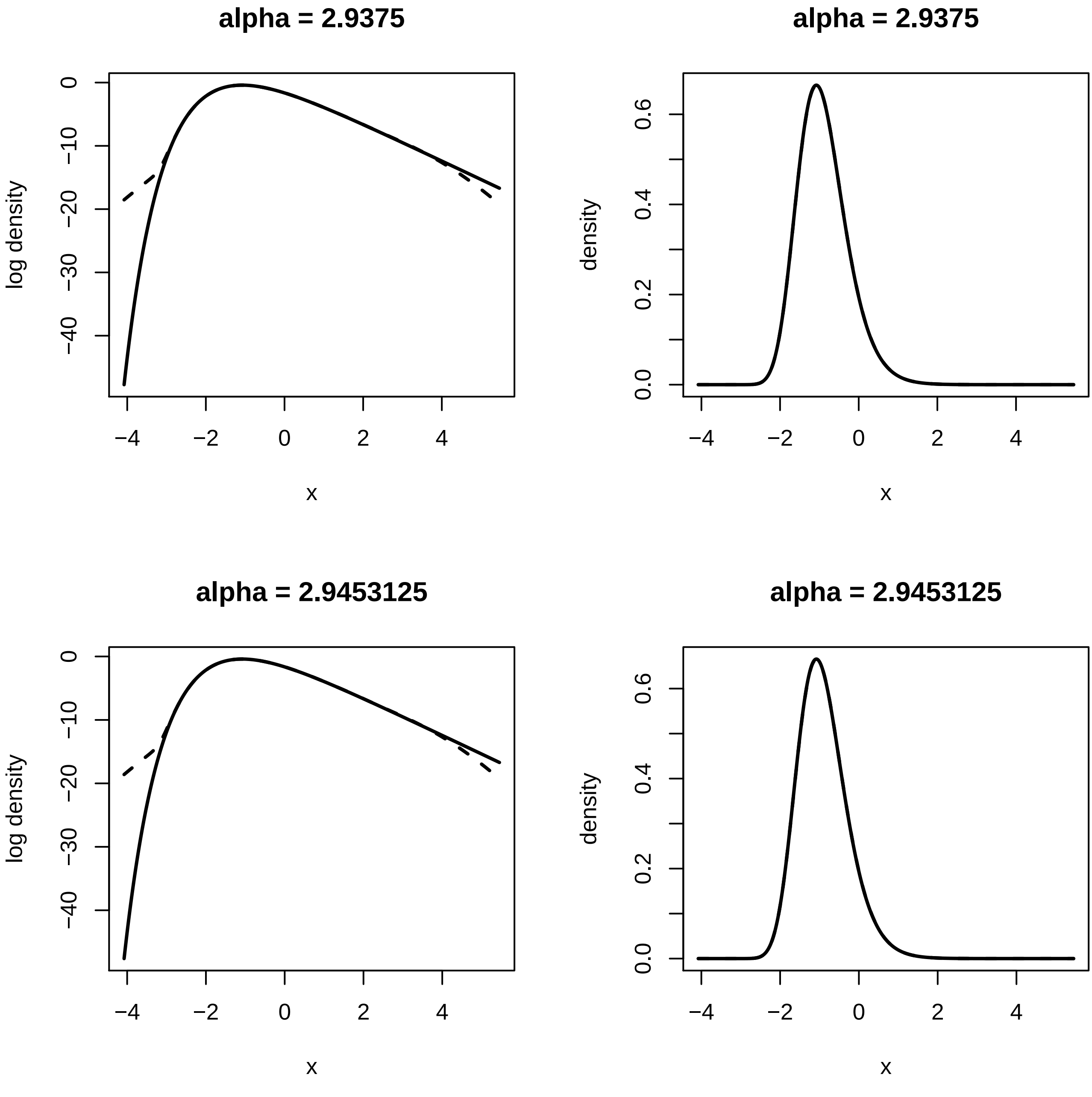

alpha = 2.875
log density
0
−10
−20
−30
−40
−50
−4
−2
0
2
4
x
alpha = 2.875
density
0.0
0.2
0.4
0.6
−4
−2
0
2
4
x
alpha = 2.8828125
log density
0
−10
−20
−30
−40
−50
−4
−2
0
2
4
x
alpha = 2.8828125
density
0.0
0.2
0.4
0.6
−4
−2
0
2
4
x

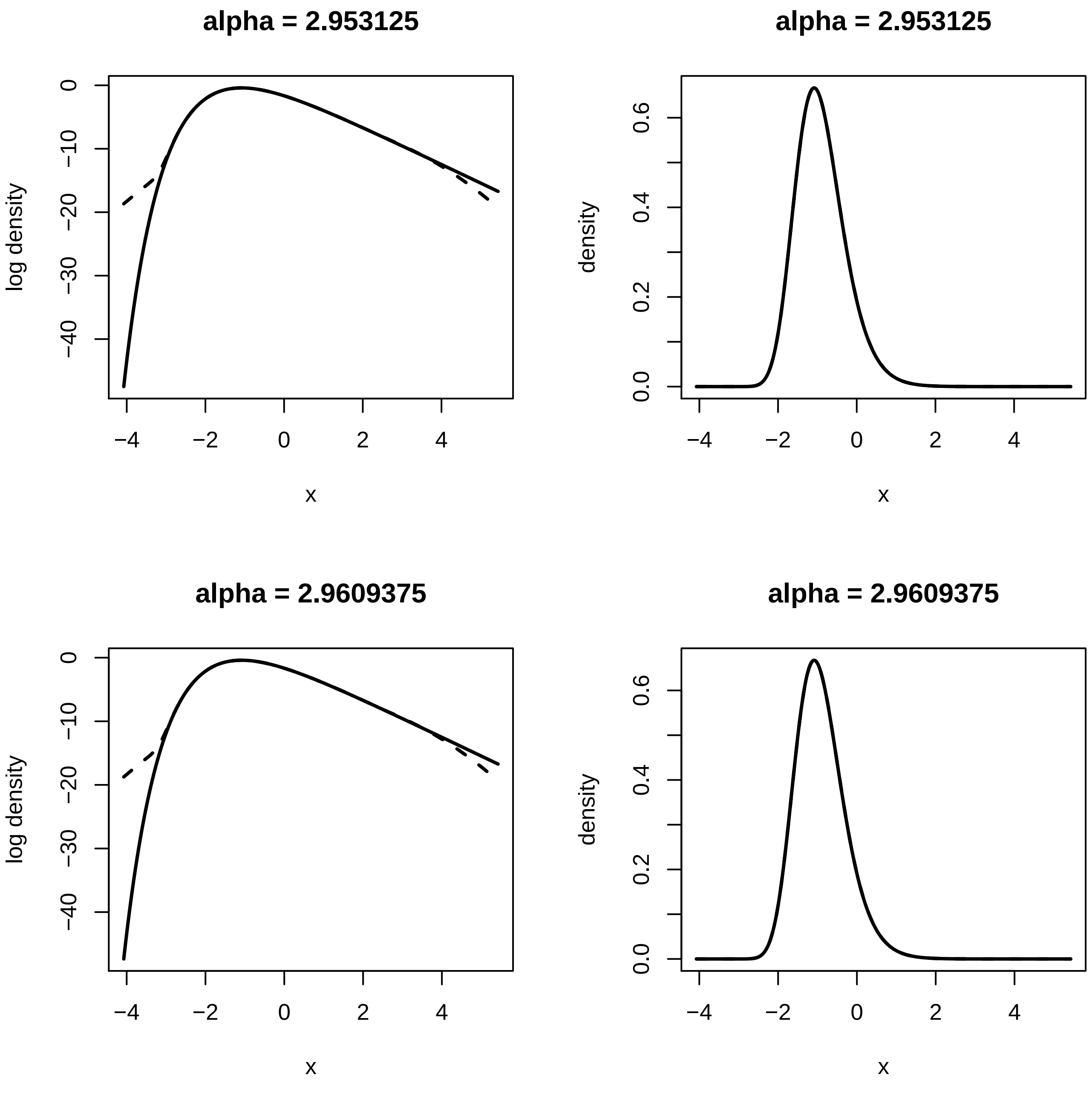

alpha = 2.890625
log density
0
−10
−20
−30
−40
−50
−4
−2
0
2
4
x
alpha = 2.890625
density
0.0
0.2
0.4
0.6
−4
−2
0
2
4
x
alpha = 2.8984375
log density
0
−10
−20
−30
−40
−50
−4
−2
0
2
4
x
alpha = 2.8984375
density
0.0
0.2
0.4
0.6
−4
−2
0
2
4
x

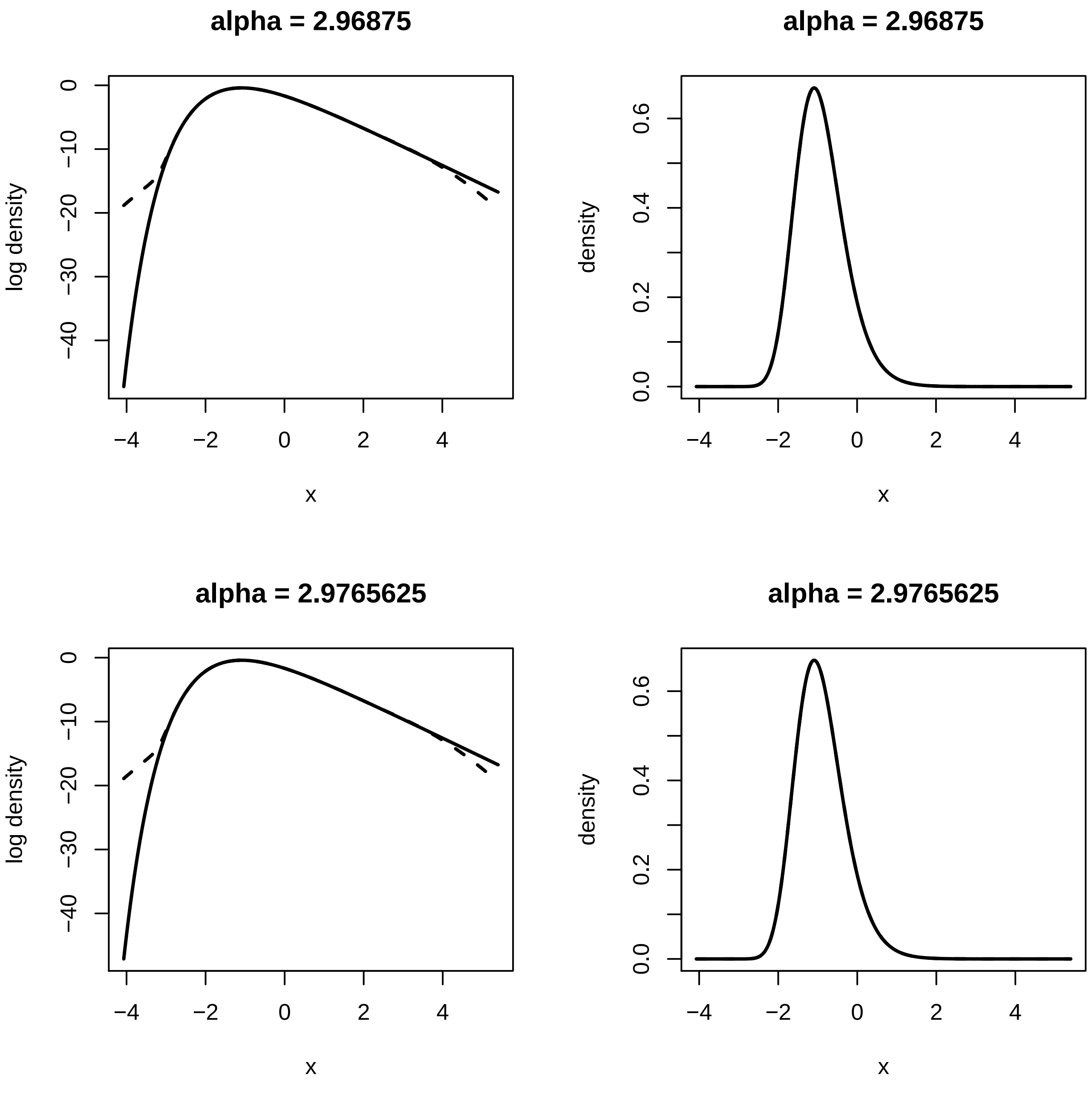

alpha = 2.90625
log density
0
−10
−20
−30
−40
−50
−4
−2
0
2
4
x
alpha = 2.90625
density
0.0
0.2
0.4
0.6
−4
−2
0
2
4
x
alpha = 2.9140625
log density
0
−10
−20
−30
−40
−50
−4
−2
0
2
4
x
alpha = 2.9140625
density
0.0
0.2
0.4
0.6
−4
−2
0
2
4
x

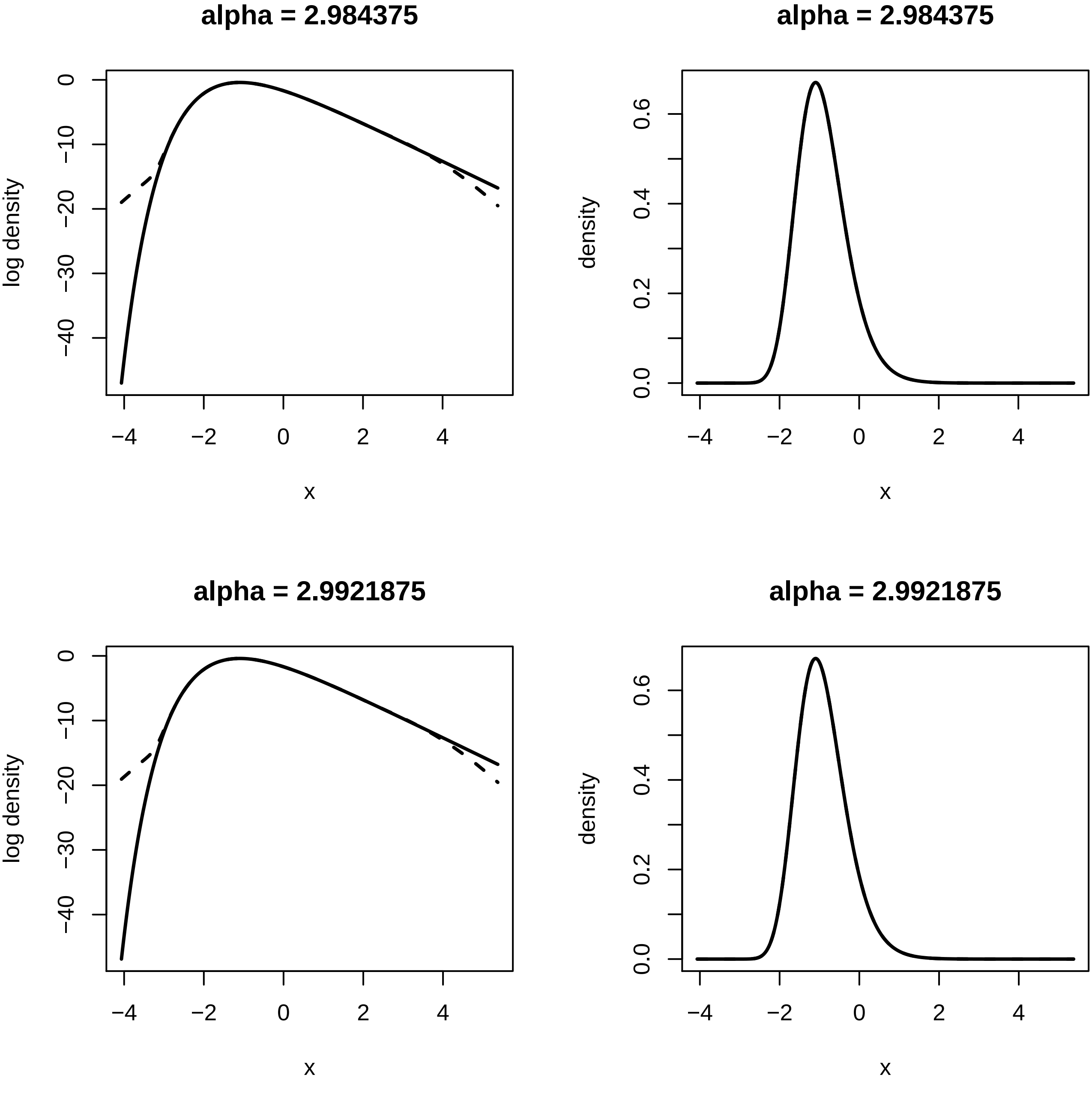


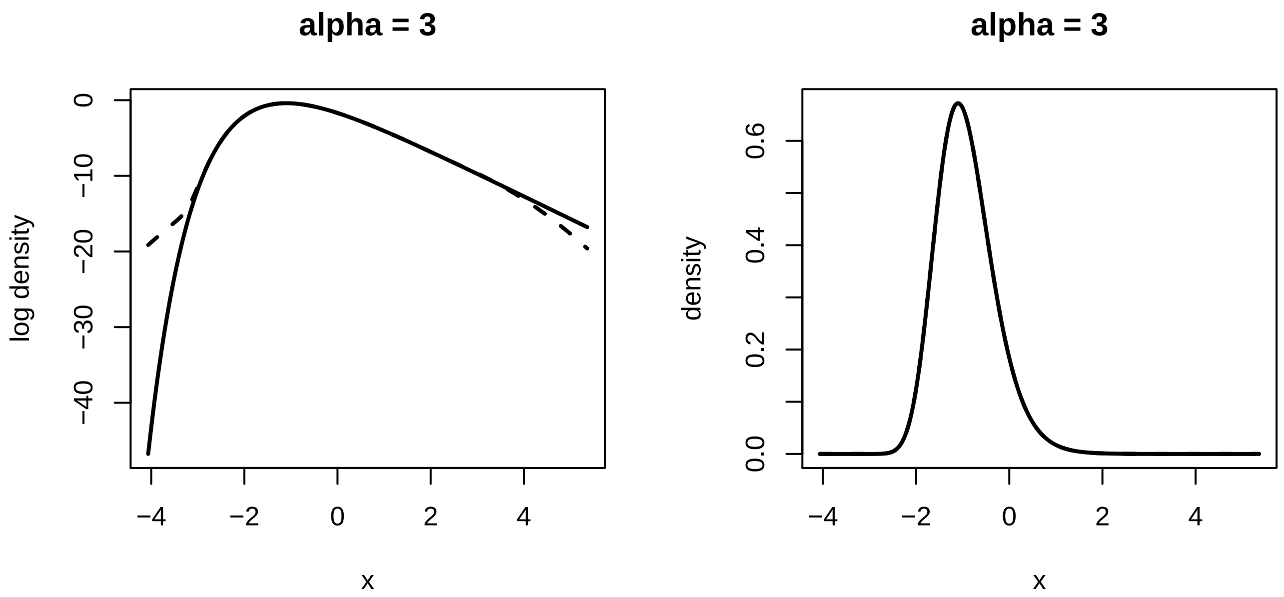
alpha = 2.921875
log density
0
−10
−20
−30
−40
−4
−2
0
2
4
x
alpha = 2.921875
density
0.0
0.2
0.4
0.6
−4
−2
0
2
4
x
alpha = 2.9296875
log density
0
−10
−20
−30
−40
−4
−2
0
2
4
x
alpha = 2.9296875
density
0.0
0.2
0.4
0.6
−4
−2
0
2
4
x


alpha = 2.9375
log density
0
−10
−20
−30
−40
−4
−2
0
2
4
x
alpha = 2.9375
density
0.0
0.2
0.4
0.6
−4
−2
0
2
4
x
alpha = 2.9453125
log density
0
−10
−20
−30
−40
−4
−2
0
2
4
x
alpha = 2.9453125
density
0.0
0.2
0.4
0.6
−4
−2
0
2
4
x


alpha = 2.953125
log density
0
−10
−20
−30
−40
−4
−2
0
2
4
x
alpha = 2.953125
density
0.0
0.2
0.4
0.6
−4
−2
0
2
4
x
alpha = 2.9609375
log density
0
−10
−20
−30
−40
−4
−2
0
2
4
x
alpha = 2.9609375
density
0.0
0.2
0.4
0.6
−4
−2
0
2
4
x


alpha = 2.96875
log density
0
−10
−20
−30
−40
−4
−2
0
2
4
x
alpha = 2.96875
density
0.0
0.2
0.4
0.6
−4
−2
0
2
4
x
alpha = 2.9765625
log density
0
−10
−20
−30
−40
−4
−2
0
2
4
x
alpha = 2.9765625
density
0.0
0.2
0.4
0.6
−4
−2
0
2
4
x


alpha = 2.984375
log density
0
−10
−20
−30
−40
−4
−2
0
2
4
x
alpha = 2.984375
density
0.0
0.2
0.4
0.6
−4
−2
0
2
4
x
alpha = 2.9921875
log density
0
−10
−20
−30
−40
−4
−2
0
2
4
x
alpha = 2.9921875
density
0.0
0.2
0.4
0.6
−4
−2
0
2
4
x



alpha = 3
log density
0
−10
−20
−30
−40
−4
−2
0
2
4
x
alpha = 3
density
0.0
0.2
0.4
0.6
−4
−2
0
2
4
x