Logistic regression

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Outline

- Odds, LOGITs and Probabilities on examples
- Simple logistic regression
  - Single binary predictor
  - Single continuous predictor
  - Interpretation of regression coefficients
- Multiple Logistic regression
  - Logistic models
  - Estimation / Inference
  - Logistic model for association test
  - Logistic model for prediction/classification
- Summary

Example 1.1: 100 participants are randomized to a new or standard treatment (50 subjects to each treatment group).

Groups	New	Standard	Total
Success	20	10	30
Failure	30	40	70
Total	50	50	100

Are chances of success equal for each treatment choice?

Example 1.1: (cont)

How to measure the chances of success?

1) The probability of success:

P_{new} = Pr (Success | new treatment) = 20/50 = 40%

P_{st} = Pr (Success | standard treatment) = 10/50 = 20%

2) The odds of success:

$O_{new} = P_{new}/(1-P_{new}) = 20/30 = 0.66$

$O_{st} = P_{st}/(1-P_{st}) = 10/40 = 0.25$

3) The natural logarithm of odds of success (also called LOGIT):

$LOGIT_{new} = \log(20/30) = -0.41$ (new treatment)

$LOGIT_{st} = \log(10/40) = \log(0.25) = -1.39$ (st. treatment)

Example 1.1: (cont)

Odds Ratio (OR) is a possible way to capture inequality in the chances of success:

- OR = O_{new}/O_{st} = (20/30)/(10/40) = 0.67/0.25=2.67
- Obviously the odds ratio is just a RATIO OF ODDS ☺ (between the new and standard treatment groups)
- If OR =1 then the success chances are the same in each group, which means $P_{new} = P_{st}$ or $O_{new}=O_{st}$.
- In our case, obviously, the odds of success are 2.67 times higher for the new treatment comparing to the standard one. If this statistically significant???

Association Test

- If chances of success are the same (no *association* between the chances of success and the chosen treatment), we would expect $O_{new}=O_{st}$, or equivalently, OR=1.
- The null hypothesis is H_0: OR=1 vs. the alternative H_a: OR≠1
- Chi-square test can be used:
  P-Val = 0.049 (significant, because < 5%)

Example 1.2a (independence): How does "no difference" in treatment success rates look? (one variant)

Groups	New	Standard	Total
Success	20	20	40
Failure	30	30	60
Total	50	50	100

In this case $P_{new} = P_{st} = 50\%$, and $O_{new}=O_{st}=1$, and OR=1

Example 1.2b (independence): How does "no difference" in success rates look? (another variant)

Groups	New	Standard	Total
Success	10	10	20
Failure	40	40	80
Total	50	50	100

In this case $P_{new} = P_{st} = 20\%$, and $O_{new}=O_{st}=0.25$, OR=1

Simple logistic regression

- The probability of success can be represented via odds or LOGITs of success.
- From Example 1.1,
  $\log(O_{new}) = -0.41$ and $\log(O_{st}) = -1.39$, so the *difference between the log odds is equal* to 0.98.
- We can combine these two log odds for different groups into one formula:

  log(odds) = -1.39 + 0.98*(treatment is new)

  (this is an example of a simple logistic regression)

Simple logistic regression (cont)

- LOGIT=log(odds) = -1.39 + 0.98*(treatment is new)
- In this logistic regression -1.39 and 0.98 are regression coefficients…
- -1.39 is called the model intercept
- 0.98 is the treatment effect

- It is important to understand the “connection” between the regression coefficients and probabilities of success

Simple logistic regression (cont)

- LOGIT = -1.39 + 0.98*(treatment is new)

- If the treatment is “standard” then
  LOGIT = -1.39 + 0.98 * 0 = -1.39 and
  odds = O_{st} = exp(-1.39)=0.25 and
  P_{st} = 20%

- If the treatment is “new” then
  LOGIT = -1.39 + 0.98 * 1 = -0.41
  odds = O_{new} = exp(-0.41) = 0.67
  P_{new} = 40%

Simple logistic regression (cont)

- If we apply antilog to 0.98 then exp(0.98)=2.67, the odds ratio!!!
- This 2.67 is different from 1, which means we have a significant increase in odds of treatment success (chi-square O-value was < 5%)

Example 2: Coronary Heart Disease (CHD)

- Risk factors for CHD include gender, age, smoking, high blood pressure, high cholesterol, obesity, etc.
- First we look at AGE as a continuous predictor

Age and CHD

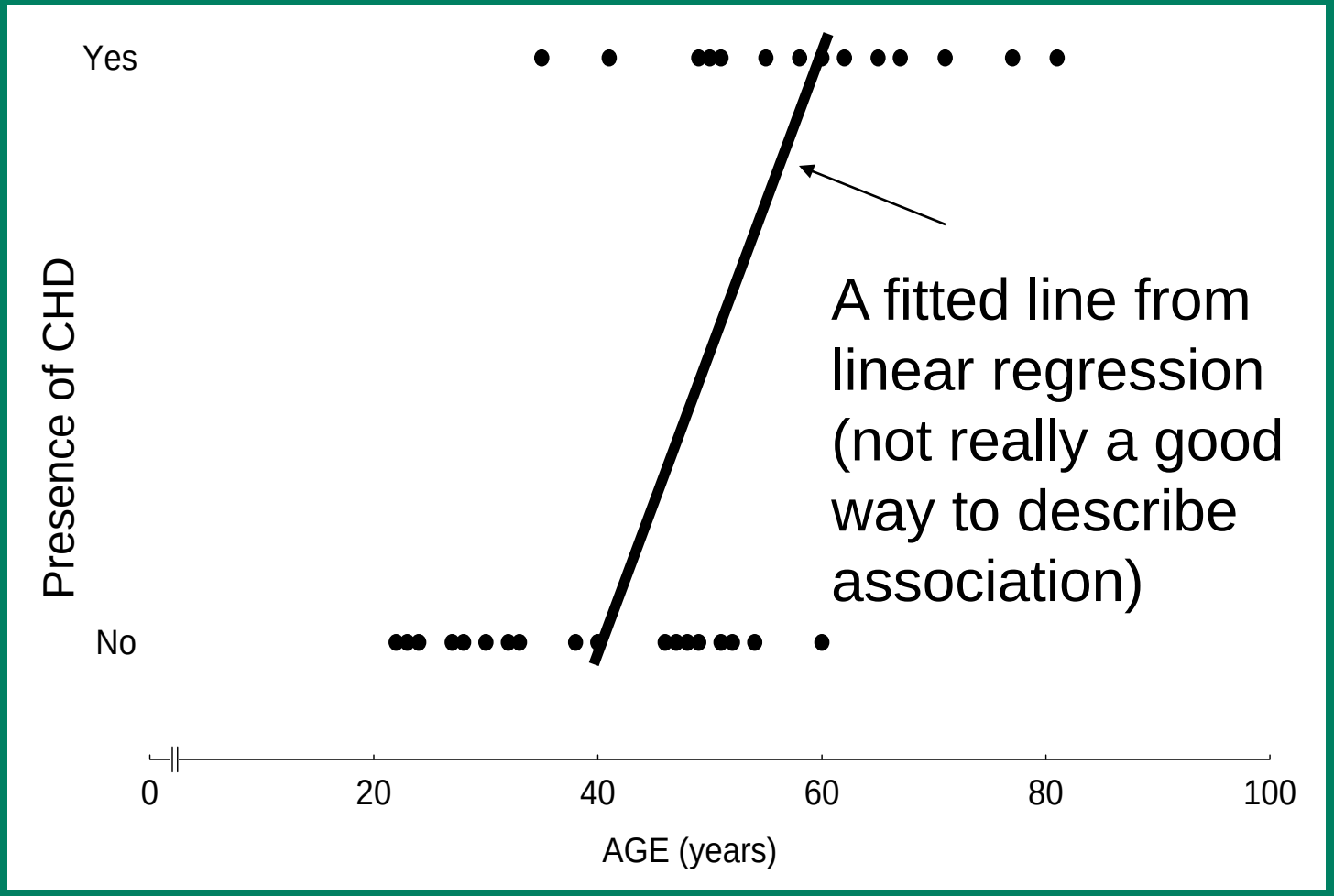
Table 1. Age and coronary heart disease

Age	CD	Age	CD	Age	CD
22	0	40	0	54	0
23	0	41	1	55	1
24	0	46	0	58	1
27	0	47	0	60	1
28	0	48	0	60	0
30	0	49	1	62	1
30	0	49	0	65	1
32	0	50	1	67	1
33	0	51	0	71	1
35	1	51	1	77	1
38	0	52	0	81	1

How to Describe Association?

- Age is a continuous variable
- Compare mean age between diseased and non-diseased groups
  - Non-diseased (38.6 years) vs. diseased (58.7 years) $\Rightarrow p<0.0001$
  - Not informative to assess the magnitude of age effect
- Look at the relationship between age and the presence of CHD

A Dot-Plot


Yes
No
Presence of CHD
A fitted line from linear regression (not really a good way to describe association)
0
20
40
60
80
100
AGE (years)

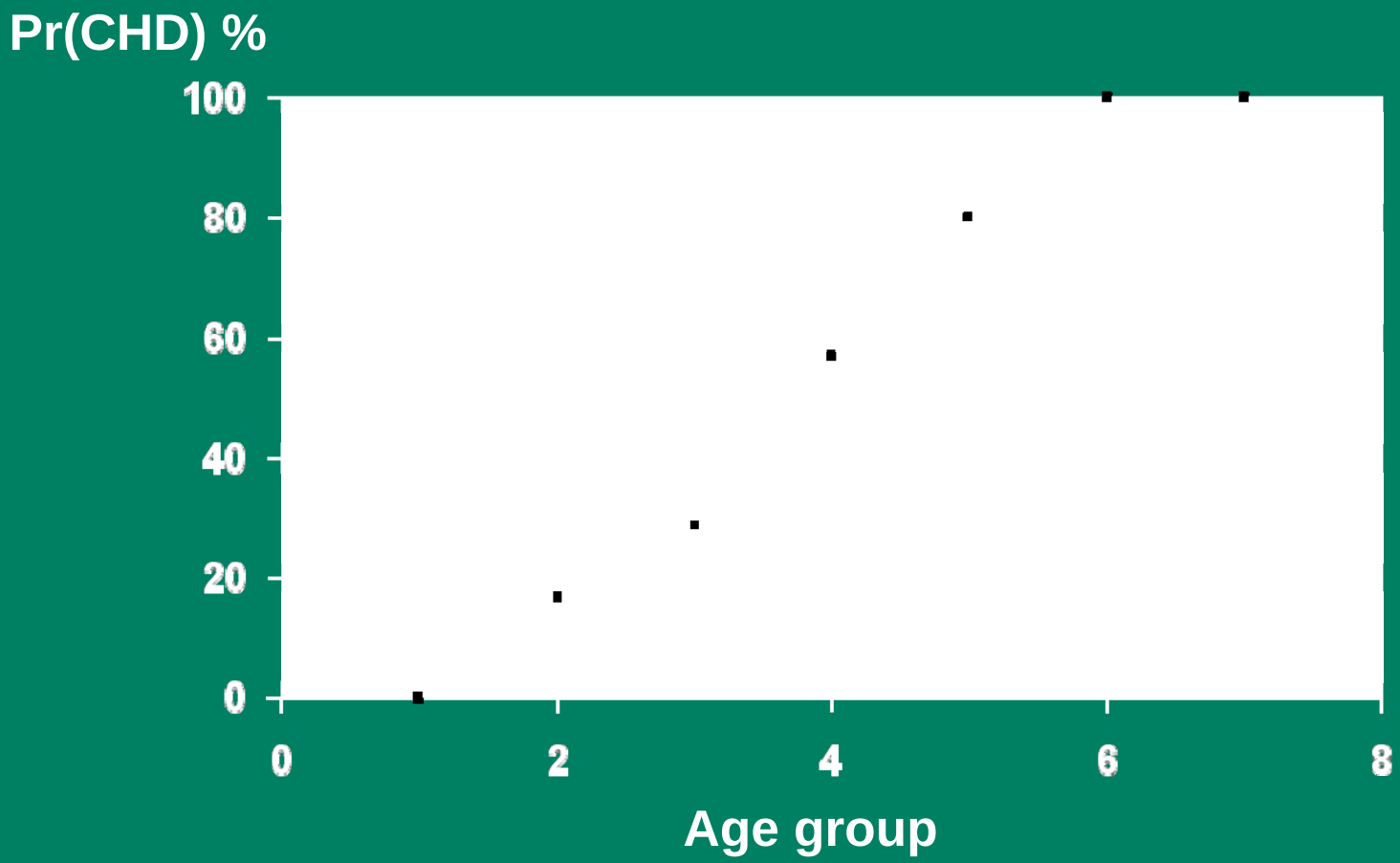
Other Options

- When the outcome variable is binary (like the presence or absence of CHD), it makes more sense to consider *probability of having the disease* at different ages
- Categorize age into multiple groups
- Look at presence/absence of disease in each age group

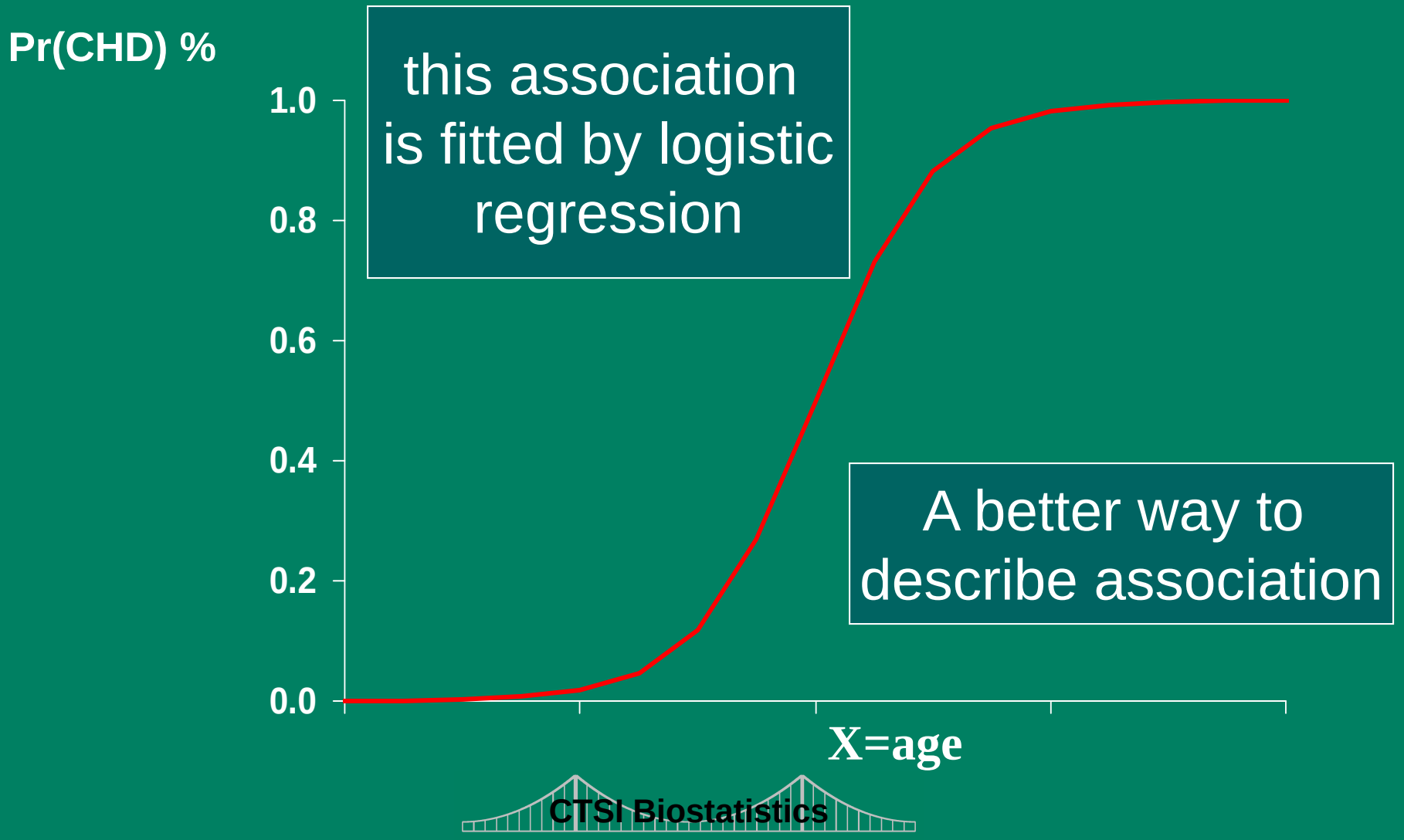
Table. Presence (%) of CHD in different age groups

Age group	# in group	Diseased	
		#	%
20 - 29	5	0	0
30 - 39	6	1	17
40 - 49	7	2	29
50 - 59	7	4	57
60 - 69	5	4	80
70 - 79	2	2	100
80 - 89	1	1	100

Dot plot of CHD presence (%) in different age groups



Logistic Curve



Interpretation of Parameters

- When I fit logistic regression (in SAS) for CHD data I have the following output:

Parameter	DF	Estimate	Standard Error	Wald Chi-Square	Pr > ChiSq
Intercept	*1*	*-6.5820*	*2.3121*	*8.1038*	*0.0044*
AGE	*1*	*0.1309*	*0.0458*	*8.1557*	*0.0043*

- This output leads to the following

  LOGIT(CHD)= -6.5820 + 0.1309*AGE

Interpretation of Parameters (cont)

AGE	LOGIT	ODDS	PROBABILITY
30	-2.65	0.07	0.07
40	-1.34	0.26	0.20
60	1.27	3.56	0.78
80	3.89	48.91	0.97

In the above ODDS = EXP(LOGIT) and PROBABILITY=ODDS/(1+ODDS)

Interpretation of Parameters (cont)

Note, exp(0.1309)=1.14 is also an odds ratio, but this odds ratio describes % increase in odds when AGE increases by 1 year.

For example,

(1) at AGE=40 the ODDS of CHD were 0.26, then at AGE=41 the ODDS=0.26*1.14=0.296,

(2) at AGE=60 the ODDS of CHD were 3.56, then at AGE=61 the ODDS=3.56*1.14=4.06,

(3) at AGE=60 the ODDS of CHD were 3.56, then at AGE=59 the ODDS=3.56/1.14=3.13

Assumptions

- Independent observations
- A linear relationship between LOGIT of CHD and AGE

What if we do not have a linear relationship between the LOGIT of CHD and AGE???

In this case we can
(only one of possible solutions)
use a categorization of AGE

AGE groups

Age and CHD

Age	CD	Age	CD	Age	CD
22	0	40	0	54	0
23	0	41	1	55	1
24	0	46	0	58	1
27	0	47	0	60	1
28	0	48	0	60	0
30	0	49	1	62	1
30	0	49	0	65	1
32	0	50	1	67	1
33	0	51	0	71	1
35	1	51	1	77	1
38	0	52	0	81	1

Two by three table (after AGE categorization)

- Frequency
  Percent
  Row Pct
  Col Pct

	20-39	40-53	54-90	Total
NO	10	6	2	18
	31.25	18.75	6.25	56.25
	55.56	33.33	11.11	
	90.91	60.00	18.18	
YES	1	4	9	14
	3.13	12.50	28.13	43.75
	7.14	28.57	64.29	
	9.09	40.00	81.82	
Total	11	10	11	32
	34.38	31.25	34.38	100.00

Odds and LOGITs for the categorized AGE

- Odds of CHD in 20-39 group = 1/10
- Odds of CHD in 40-53 group = 4/6
- Odds of CHD in 54-90 group = 2/2

- LOGIT of CHD in 20-39 group = log(1/10)
- LOGIT of CHD in 40-53 group = log(4/6)
- LOGIT of CHD in 54-90 group = log(2/2)

Odds ratios for AGE Categorization

- Odds ratio of CHD may change over age. A more complete data actually shows
  - OR=6.7=(10*4)/(6*1) for ‘40-53’ vs. ’20-39’
  - OR=45=(10*9)/(2*1)) for ‘54-90’ vs. ’20-39’
- When we categorize ‘age’ into multiple groups each of the age intervals should have a reasonable number of observations (not true in out case !!! Which makes the results unreliable)… some researchers require cell counts to be at least 5 !!!

Multiple Logistic Regression

- "Multiple" means more than 1 predictor in the model; "simple" means single predictor
- Three groups:
  age$\leq$39, 39<age $\leq$53, age>53
- Define:
  - age_1=1, for 39<age $\leq$53; 0, otherwise
  - age_2=1, for age>53; 0, otherwise.
- age$\leq$39 is the baseline group.

Multiple Logistic Regression (SAS output)

- **Analysis of Maximum Likelihood Estimates**

Parameter	DF	Estimate	Standard Error	Wald Chi-Square	Pr > ChiSq
Intercept	1	-2.3026	1.0488	4.8199	0.0281
age1	1	1.8971	1.2315	2.3730	0.1235
age2	1	3.8066	1.3081	8.4686	0.0036

This output leads to the following:

LOGIT = -2.3026 + 1.8971*(39<age<=53) + 3.8066*(age > 53)

Interpretation of Parameters (categorical AGE)

AGE Cat	LOGIT	ODDS	PROBABILITY
20-39	-2.3036	0.0999	0.0908
40-53	-0.4055	0.6666	0.4000
54-90	1.5040	4.4997	0.8182

Compare the above results with the "continuous" AGE case

Further Improvement

- We have seen how two predictors can be incorporated in the model
- Risk factors for CD include gender, age, smoking, high blood pressure, high cholesterol, obesity, etc.
- A model including multiple risk factors
  - Adjusts for other risk factors
  - Provides better prediction

Example 3: low birth weight data

(Hosmer & Lemeshow "Applied Logistic

Goal: to identify risk factors associated with lower birth weight (variable "low")

Dataset: 189 women (59 lower birth weight babies, and 130 – normal weight babies)

Possible Risk Factors: age ("AGE"), subject's weight ("LWT"), race ("race2" and "race3"), and the number of physician visits ("FTV")

The SAS output (all four predictors):

Analysis of Maximum Likelihood Estimates

Parameter	Estimate	Standard Error	Chi-Square	Pr >ChiSq
Intercept	**1.2953**	**1.0714**	**1.4616**	**0.2267**
AGE	**-0.0238**	**0.0337**	**0.4988**	**0.4800**
LWT	**-0.0142**	**0.0065**	**4.7428**	**0.0294**
race2	**1.0039**	**0.4979**	**4.0660**	**0.0438**
race3	**0.4331**	**0.3622**	**1.4296**	**0.2318**
FTV	**-0.0493**	**0.1672**	**0.0869**	**0.7681**

Here “race2” and “race3” are indicators that RACE=2 and RACE=3 (the race categories were enumerated in the data as 1, 2 and 3)

The SAS output (excluding AGE and FTV):

Parameter	Estimate	Standard Error	Chi-Square	Pr > ChiSq
Intercept	**0.8057**	**0.8452**	**0.9088**	**0.3404**
LWT	**-0.0152**	**0.00644**	**5.5886**	**0.0181**
race2	**1.0811**	**0.4881**	**4.9065**	**0.0268**
race3	**0.4806**	**0.3567**	**1.8156**	**0.1778**

There are many approaches to model selection and excluding insignificant predictors is only one of them

Model equation:
LOGIT(Low birth weight) = 0.8057 – 0.01*LWT + 1.0811*race2+0.4806*race3

Adjusted Odds Ratios

- The effect of LWT (last weight) is described by the regression coefficient of **-0.0152… taking antilog we obtain the adjusted (for race) odds ratio exp(-0.0152)=0.98**
- **Interpretation: the odds of lower birth weight decrease (by 2%) with one lb increase in LWT given other predictors (race) stay the same**

Multiple Logistic Regression Objectives:

- To find significant predictors (risk or protective factors)
- To build a predictive model for predicting the LOGIT
- To control for effect of significant predictors (risk factors)… a way to eliminate confounding effects

Prediction & Classification

- Prediction: From the fitted model, a predicted *probability* can be computed for each set of predictors
- Classification: If the predicted probability exceeds some cut-off point, the observation is predicted to be an *event* observation; otherwise, it is predicted as a *nonevent*.

Logistic Regression Assumptions

- Independent observations
- Linear relationship between the log of odds and continuous covariates
- Constant odds ratios across values of continuous predictors (if not, categorize them !)

Summary

Logistic regression

- deals with binary outcomes

- allows multiple predictor variables, which can be continuous, categorical or ordinal

- provides estimates of adjusted odds ratios

References

- Regression methods in Biostatistics: Linear, Logistic, Survival and Repeated Measures Models, Eric Vittinghoff et al., Springer, 2005.
- Applied Logistic Regression, Hosmer & Lemeshow, Wiley, 1989
- Categorical Data Analysis using the SAS system, 2ndEd, by Stokes, et al., SAS Institute INC, 2000.

Resources

- The Clinical and Translation Science Institute (CTSI) supports education, collaboration, and research in clinical and translational science: www.ctsi.mcw.edu
- The Biostatistics Consulting Service provides comprehensive statistical support

  http://www.mcw.edu/biostatsconsult.htm

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  - Tuesday 9 – 5 PM
  - School of Nursing, Clark Hall