SURVIVAL DISTRIBUTIONS AND THEIR CHARACTERISTICS

A CONTRIBUTION TO
THE ENCYCLOPEDIA OF BIOSTATISTICS

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INTRODUCTION

Many applications in biostatistics involve the modeling of lifetime data. In these applications the outcome of interest is the time T, until some event occurs. This event may be death, the appearance of a tumor, the development of some disease, recurrence of a disease, conception, cessation of smoking, and so forth. Here T is a non-negative random variable from a homogenous population.

In this article we shall examine how the distribution of T can be characterized. Four functions characterize the distribution of T, namely, the *survival function*, which is the probability of an individual surviving beyond time t, the *hazard rate* which is approximately the chance an individual of age t experiences the event in the next instant in time, the *probability density (or mass) function*, which is the approximate unconditional probability of the event occurring at time t, and the *mean residual life* at time t, which is the mean time to the event of interest, given the event has not occurred at t. If we know any one of these four functions, then the other three can be uniquely determined. These functions are introduced for continuous, discrete and mixed random variable in the following sections and the interrelationship among the four functions are discussed.

The distribution of the time to an event can also be characterized by the aging properties of the distribution of T. Aging classes are based on certain properties of one of the four basic quantities that describe the distribution of T. These classes are defined and some basic properties of these classes are discussed in the final section.

THE SURVIVAL FUNCTION

The basic quantity employed to describe time-to-event phenomena is the survival function. This function, also known as the survivor function or survivorship function, is the probability an individual survives beyond time t. It is defined as

$$S(t) = Pr\left[T \geq t\right].$$

In the context of equipment or manufactured item failures, $S(t)$ is referred to as the reliability function. Note that the survival function is a non increasing function with a value of 1 at the origin and 0 as t approaches infinity.

If T is a continuous random variable then $S(t)$ is a continuous monotone decreasing function and the survival function is the complement of the cumulative distribution function $F(t) = Pr\left[T \leq t\right]$. That is $S(t) = 1 - F(t)$. The survival function is the integral of the probability density function $f(t)$. That is,

$$S(t) = Pr\left(T \geq t\right) = \int_{t}^{\infty} f(u)du$$

Thus, we have the following relationship:

$$f(t) = -\frac{dS(t)}{dt}.$$

Note that $f(t)\Delta t$ may be thought of as the "approximate" probability of the event occurring at time t and that $f(x)$ is a non-negative function with the area under $f(x)$ being equal to one.

Example

A common distribution used in many applications in the Weibull distribution with probability density function $f(t) = \lambda\alpha t^{\alpha-1}\exp(-\lambda t^{\alpha}), \lambda > 0, \alpha > 0$. The exponential distribution is a special case of the Weibull distribution when $\alpha = 1$. The survival function for the Weibull distribution is $S(t) = \exp(-\lambda t^{\alpha}), \lambda > 0, \alpha > 0$. Survival curves with a common median of 6.93 are exhibited in Figure 1 for $\lambda = .26328,\ \alpha = .5;\ \lambda = .1,\ \alpha = 1$; and $\lambda = .00208, \alpha = 3$. $\diamondsuit$

When T is a discrete random variable then the survival function is a non increasing left-continuous step function. If T can take on values $t_0 < t_1 < t_2 < \ldots$ with probability mass function (p.m.f.) $p(t_j) = \Pr(T = t_j), j = 1, 2, \ldots$ then

$$S(t) = Pr(X \geq t) = \sum_{j:t_j \geq t} p(t_j).$$

Note that the survival function and probability mass function are related by

$$p(t_j) = S(t_j) - S(t_{j+1})$$

. Here we have defined $S(t) = Pr[T \geq t]$ as was the case in [3] and [4]. This definition was used to make later formulas for the discrete case simpler. Other authors (c.f. [5] and [6]) have defined $S(t) = Pr[T > t]$ which makes the relationship $S(t) = 1 - F(t)$ hold for both the discrete and continuous case.

THE HAZARD FUNCTION

A basic quantity, foundational in survival analysis, is the hazard function. This function is also known as the conditional failure rate in reliability, the force of mortality in demography, the age-specific failure rate in epidemiology, the inverse of the Mill's ratio in economics or simply as the hazard rate. The hazard rate is defined as

$$h\left(t\right) = \lim_{\Delta t \to 0} \frac{Pr\left[t \leq T < t + \Delta t | T \geq t\right]}{\Delta t}. \tag{1}$$

The hazard rate is a non-negative function. It tells us how quickly individuals of a given age are experiencing the event of interest. The quantity $h\left(t\right)\Delta t$ is the approximate probability that an individual who has survived to age t will experience the event in the interval $\left(t,\, t + \Delta t\right)$.

This function is particularly useful in determining the appropriate failure distributions utilizing qualitative information about the mechanism of failure and for describing the way in which the chance of experiencing the event changes with time. There are many general shapes for the hazard rate. Some generic types of hazard rates are increasing, decreasing, constant, bathtub-shaped or hump-shaped hazard rates. Models with increasing hazard rates arise when there is natural aging or wear-out. Decreasing hazard functions are much less common but find occasional use when there is a very early likelihood of failure such as in certain types of electronic devices or in patients experiencing certain types of transplants. Decreasing hazard rates often arise as models for heterogenous populations where the hazard rates of members of the population are random (See Frailty models). Most often a bathtub-shaped hazard is appropriate in populations followed from birth. Most population mortality data follows this type of hazard function where, during an early period, deaths result primarily from infant diseases after which the death rate stabilizes followed by an increasing hazard rate due to the natural aging process. Finally, if the hazard rate is increasing early and eventually begins declining, then the hazard is termed hump-shaped. This type of hazard rate is often used in modeling survival after successful surgery where there is an initial increase in risk due to infection, hemorrhaging, or other complications just after the procedure, followed by a steady decline in risk as the patient recovers.

If T is a continuous random variable, then

$$h(t) = f(t)/S(t) = -\frac{d\ln[S(t)]}{dt}$$

A related quantity is the cumulative hazard function $H(t)$, defined by

$$H(t) = \int_0^t h(u)du = -\ln[S(t)].$$

Thus for continuous lifetimes we have the following relationship:

$$S(t) = \exp\{-H(t)\} = \exp\left\{-\int_0^t h(u)du\right\}.$$

One particular distribution, which is flexible enough to accommodate increasing $(\alpha > 1)$, decreasing $(\alpha < 1)$,or constant hazard rates $(\alpha = 1)$, is the Weibull distribution. Hazard rates, $\mathrm{h}(x) = \alpha\ \lambda\ x^{\alpha-1}$, are plotted in Figure 2 for the Weibull distribution with $\lambda = .26328, \alpha = .5; \lambda = .1, \alpha = 1$; and $\lambda = .00208, \alpha = 3$. One can see that, though the three survival functions have the same basic shape, the three hazard functions are dramatically different.$\diamondsuit$

When T is a discrete random variable, the hazard function is

$$h(t_j) = \Pr(T = t_j | T \geq t_j) = \frac{p(t_j)}{S(t_j)},\ j = 1, 2, ...$$

Since $p(t_j) = S(t_j) - S(t_{j+1})$ we have

$$h(t_j) = 1 - S(t_{j+1}) / S(t_j),\ j = 1, 2, ...$$

so that the survival function is related to the hazard function by

$$S(t) = \prod_{j:t_j<t} [1 - h(x_j)].$$

For discrete lifetimes the "cumulative hazard" function is defined by

$$H(t) = \sum_{j:t_j<t} h(t_j). \tag{2}$$

Notice that for this definition the relationship $S(t) = \exp[-H(t)]$ no longer holds true. Some authors (Cox and Oakes [3]) prefer to define the cumulative hazard for discrete lifetimes, as

$$H(t) = \sum_{t_j < t} \ln[1 - h(t_j)], \tag{3}$$

Note that for this definition the relationship for continuous lifetimes, $S(t) = \exp[-H(t)]$ will then be preserved for discrete lifetimes. If the $h(t_j)$ are small, (2) will be a first order approximation to (3).

The hazard rate is well-defined quantity for the case where T has both discrete and continuous components. In this case the hazard function defined by (1) will have a continuous part, $h_c(t)$ and a discrete part with mass h_j at time $t_1 < t_2 < \ldots$. The survival function in this case can be expressed as

$$S(t) = \exp\left\{-\int_0^t h_c(u)du\right\} \prod_{j:t_j<t} (1 - h_j)$$

For any survival function one can express the relationship between the hazard rate and the survival function by the using the notion of a product integral. For a function, G(), define the product integral of $1 - dG(u)$ over the range a to b by

$$P_a^b[1 - dG(u)] = \lim \prod_{k=1}^{r} \{1 - [G(u_k) - G(u_{k-1})]\},$$

where $a = u_1 < \ldots < u_r = b$ and the limit is taken as $r \to \infty$ and $u_k - u_{k-1} \to 0$. Here G is a function of locally bounded variation which is continuous from the right and have finite left hand limits. If we define the cumulative hazard rate as

$$H(t) = \int_0^t h_c(u)du + \sum_{j:t_j<t} h_j$$

then the survival function in the continuous, discrete or mixed case is given by

$$S(t) = P_0^t \left[1 - dH(u)\right].$$

Because of this property the product integral plays an important role in survival analytic techniques.

THE MEAN RESIDUAL LIFE FUNCTION

The fourth basic parameter of interest is the mean residual life at time t. This parameter measures, for individuals of age t, their expected remaining lifetime. It is defined as

$$mrl(t) = E(T - t|T \geq t).$$

It can be shown, using integration by parts or a partial summation formula, that the mean residual life is the area under the survival curve to the right of t divided by $S(t)$. Note that the mean life, $\mu = mrl(0)$, is the total area under the survival curve.

For a continuous random variable we have

$$mrl(t) = \frac{\int_t^\infty (u-t)f(t)du}{S(t)} = \frac{\int_t^\infty S(u)\,dt}{S(t)}$$

and

$$\mu = E(T) = \int_0^\infty uf(u)du = \int_0^\infty S(u)du.$$

Also the variance of T is related to the survival function by

$$Var(T) = 2\int_0^\infty uS(u)du - \left[\int_0^\infty S(u)du\right]^2.$$

In some applications the median residual life, rather then the mean residual life is of interest. To define this quantity recall that the 100pth percentile of a random variable X with cumulative distribution function (survival function) $F(x)$ $(S(x))$ is the value x_p such that

$$F(x_p) \geq p \quad \text{and} \quad S(x_p) \geq 1-p.$$

The median lifetime is the 50th percentile, $x_{.5}$, of the distribution of X. If X is a continuous random variable then the pth quantile is found by solving the equation $S(x_p) = 1 - p$. It follows that the median lifetime, for a continuous random variable X, is the value $x_{.5}$ such that

$$S(x_{.5}) = 0.5.$$

The median residual life time of T at time t, $mdrl(t)$, is defined as the median time to the event for an individual who has survived to time t. That is, $mdrl(t)$ is solution to the equation

$$\frac{S(mdrl(t))}{S(t)} = .5.$$

The population median is simply the median residual life at time 0.

To illustrate these quantities consider the three Weibull distributions considered earlier. Figure 3 shows the mean residual life function for the Weibull models with $\alpha = 0.5, 1.0$ and 3.0. As the figure shows the mean residual life is constant for the exponential distribution ($\alpha = 1$), decreasing for the case where $\alpha = 3$ and increasing for the case where $\alpha = 0.5$. Note that the trend in the mean residual life is reversed from the trend in the hazard rate in that when the hazard rate is increasing, reflecting aging, the mean residual life is decreasing. Figure 4 depicts the median residual life functions for the three Weibull models. The shapes of the functions are quite similar to the shape of the mean residual life functions.

RELATIONSHIP BETWEEN CHARACTERIZATIONS

Interrelationships between the characterizations discussed earlier, for a continuous lifetime T, may be summarized as follows:

$$S(t) = \int_t^\infty f(u)du$$

$$
\begin{aligned}
&= exp\left\{-\int_0^t h(u)du\right\} \\
&= exp\left\{-H(t)\right\} \\
&= \frac{mrl(0)}{mrl(t)} \exp\left\{-\int_0^t \frac{du}{mrl(u)}\right\};
\end{aligned}
$$

$$
\begin{aligned}
f(t) &= -\frac{d}{dt}S(t) \\
&= h(t)S(t) \\
&= \left(\frac{d}{dt}mrl(t)+1\right)\left(\frac{mrl(0)}{mrl(t)^2}\right)\exp\left\{-\int_0^t \frac{du}{mrl(u)}\right\}
\end{aligned}
$$

$$
\begin{aligned}
h(t) &= -\frac{d}{dt}\ln[S(t)] \\
&= \frac{f(t)}{S(t)} \\
&= \left(\frac{d}{dt}mrl(t)+1\right)/mrl(t);
\end{aligned}
$$

and

$$
\begin{aligned}
mrl(t) &= \frac{\int_t^\infty S(u)du}{S(t)} \\
&= \frac{\int_t^\infty (u-t)f(u)du}{S(t)}
\end{aligned}
$$

For a discrete random variable we have the following relationships:

$$
\begin{aligned}
S(t) &= \sum_{j:t_j \geq t} p(t_j) \\
&= \prod_{j:t_j < t} [1 - h(t_j)].
\end{aligned}
$$

If T is an integer valued random variable with mean residual life at time k equal to m_k, $k = 0, 1, 2, ...$and m_0 is finite then we have

$$
S(k) = \frac{1 + m_0}{m_k} \prod_{j=0}^{k} \frac{m_j}{1 + m_j}.
$$

Also, for any discrete survival function, we have

$$
\begin{aligned}
p(t_j) &= S(t_j) - S(t_{j+1}) \\
&= h(t_j) S(t_j), j = 1, 2, ...; \\
h(t_j) &= \frac{p(t_j)}{S(t_j)},
\end{aligned}
$$

and

$$
mrl(t) = \frac{[t_{k+1} - t]\, S(t_{k+1}) + \sum_{j:t_j \geq t_{k+1}} [t_{j+1} - t_j] S(t_{j+1})}{S(t)}, \text{ for } t_k \leq t < t_{k+1}
$$

CLASSES OF AGING DISTRIBUTIONS

An important characteristic of survival distribution is its aging properties. There are a number of classes that have been suggested in the literature to categorize distributions based on their aging properties or their dual. The first aging class is the class of increasing hazard rate (IHR) distributions and the dual class of decreasing hazard rate (DHR) distributions. A survival distribution is said to be in the IHR (DHR) class if and only if

$$\frac{S(t+x)}{S(t)} = S(xlt) \text{ is decreasing (increasing) in } t \text{ for all } x.$$

The definition says that the T has the IHR aging property if the probability an individual of age t survives an addition x period of time is decreasing with time. If T is a continuous random variable then an equivalent definition of the IHR (DFR) class is that the hazard rate $h(t)$ is increasing (decreasing) for all t. Examples of distributions that fall in the IHR class are the Weibull distribution with $\alpha > 1$ and the gamma distribution with shape parameter greater than one.

A second, more general aging class is the class of increasing (decreasing) hazard rate on the average, IHRA (DHRA), distributions. A distribution is said to fall in the IHRA (DHRA) class if and only if

$$-\left(\frac{1}{t}\right)\ln\left[S(t)\right] \quad \text{is increasing (decreasing) in } t. \tag{4}$$

The definition arises by declaring a distribution to be in the IHRA class when its cumulative hazard rate, -ln $[S(t)]$ is increasing faster than the cumulative hazard rate of an exponential random variable, t. Since the exponential distribution reflects a model with no aging, this class is one of distributions for which individuals are, on the average, aging. There are several equivalent definitions of a IHRA class. Since (4) implies that $S^{1/t}(t)$ is increasing in t we have that T is in the IHRA class if and only if $S(\theta t) \geq S^{\theta}(t)$. A second characterization of the IHRA class is that if T is in the IHRA class then for any $\lambda > 0$ the quantity $S(t) - e^{-\lambda t}$ has at most one change of sign and if it does have a change in sign then it is from + to −. The class of IHRA distributions is larger than the class of IHR distributions in that every IHR distribution is an IHRA distribution but the converse is not true.

A third aging class is the class of decreasing (increasing) mean residual life, DMRL (IMRL) distributions. A distribution is said to be in the DMRL (IMRL) class if

$$mrl(t) = \frac{\int_t^{\infty} S(x)dx}{S(t)} \text{ is decreasing (increasing) in } t.$$

This aging class, which include all IHR models, is one where the mean remaining life of an individual of age t is becoming shorter as t increases.

A fourth aging class is the class of new better (worse) than used NBU (NWU) distributions. Here a distribution is in the NBU (NWU) class if and only if

$$S(x+t) \leq (\geq)\, S(x)S(t) \text{ for any } x \text{ and } t.$$

An equivalent definition for the NBU class is

$$\frac{S(x+t)}{S(t)} = Pr\,[T \geq x+t | T \geq t] \leq Pr\,[T \geq x] = S(x).$$

From this second definition we see that T has an NBU distribution if the probability an individual of age t lives an additional x time units is smaller than the probability an individual of age 0 survives to age x. This aging class includes all the IHRA distributions.

A fifth aging class is the class of new better (worse) that new in expectation, NBUE (NWUE) distributions. A distribution is in the NBUE (NWUE) class if its mean, μ, is finite and

$$\int_t^\infty S(u)du \leq (\geq)\, \mu S(t) \text{ for all } t.$$

The NBUE class is one where the mean residual life of an individual of age t is less that the mean of an individual of age 0.

A final aging class is the class of harmonic new better (worse) than used in expectation, HNBUE (HNWUE) distributions. A distribution is said to be in the HNBUE (HNWUE) class if its mean is finite and

$$\int_t^\infty S(u)du \leq \mu \exp(-t/\mu).$$

An equivalent definition for the HNBUE class is

$$\left\{ \frac{1}{t} \int_0^t \frac{dx}{mrl(x)} \right\}^{-1} \leq mrl(0).$$

This means that for a HNBUE distribution the integral harmonic value of the residual life of an individual of age t is smaller than the same quantity for a newly born individual.

The aging classes are ordered as follows:

$IHR \Longrightarrow IHRA \Longrightarrow NBU \Longrightarrow NBUE \Longrightarrow HNBUE$

$IHR \Longrightarrow DMRL \Longrightarrow NBUE \Longrightarrow HNBUE$

$DHR \Longrightarrow DHRA \Longrightarrow NWU \Longrightarrow NWUE \Longrightarrow HNWUE$

$DHR \Longrightarrow IMRL \Longrightarrow NWUE \Longrightarrow HNWUE$

Further discussion of these failure classes can be found in Barlow and Proschan [1] and Basu and Ebrahimi [2].

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Figure 1
Comparison of Weibull Survival Functions

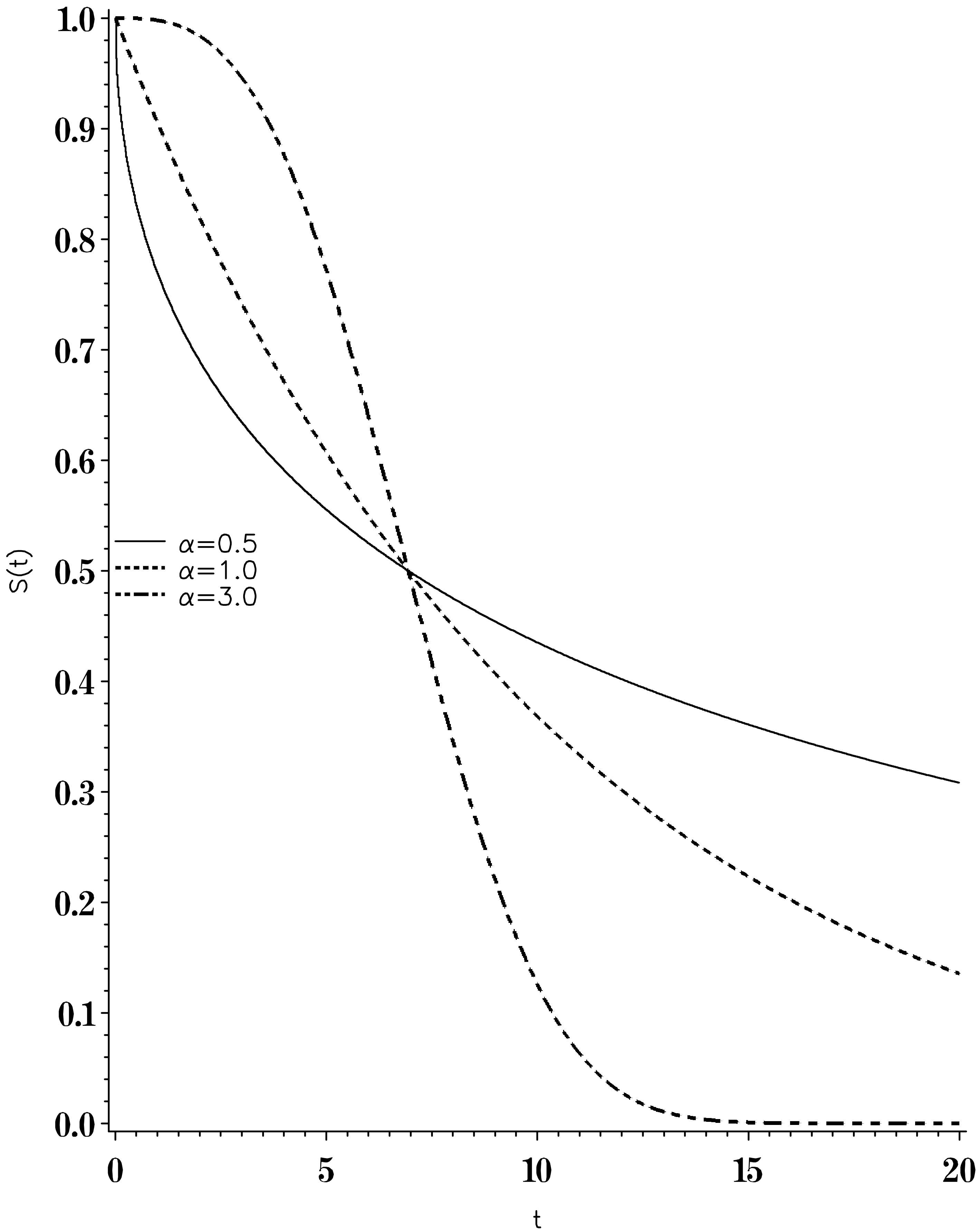

Figure 2
Comparison of Weibull Hazard Functions

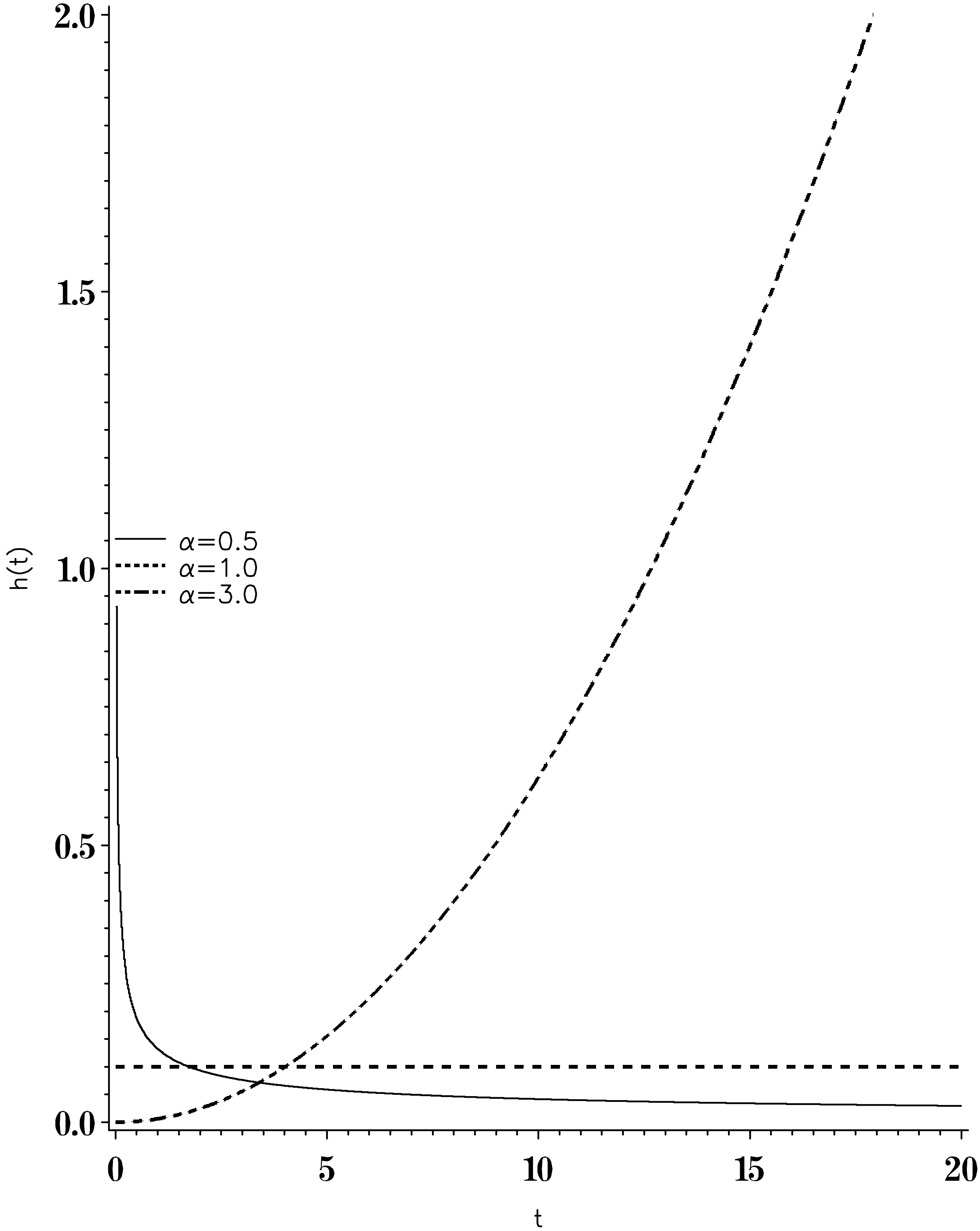

Figure 3
Comparison of Weibull Mean Residual Life Functions

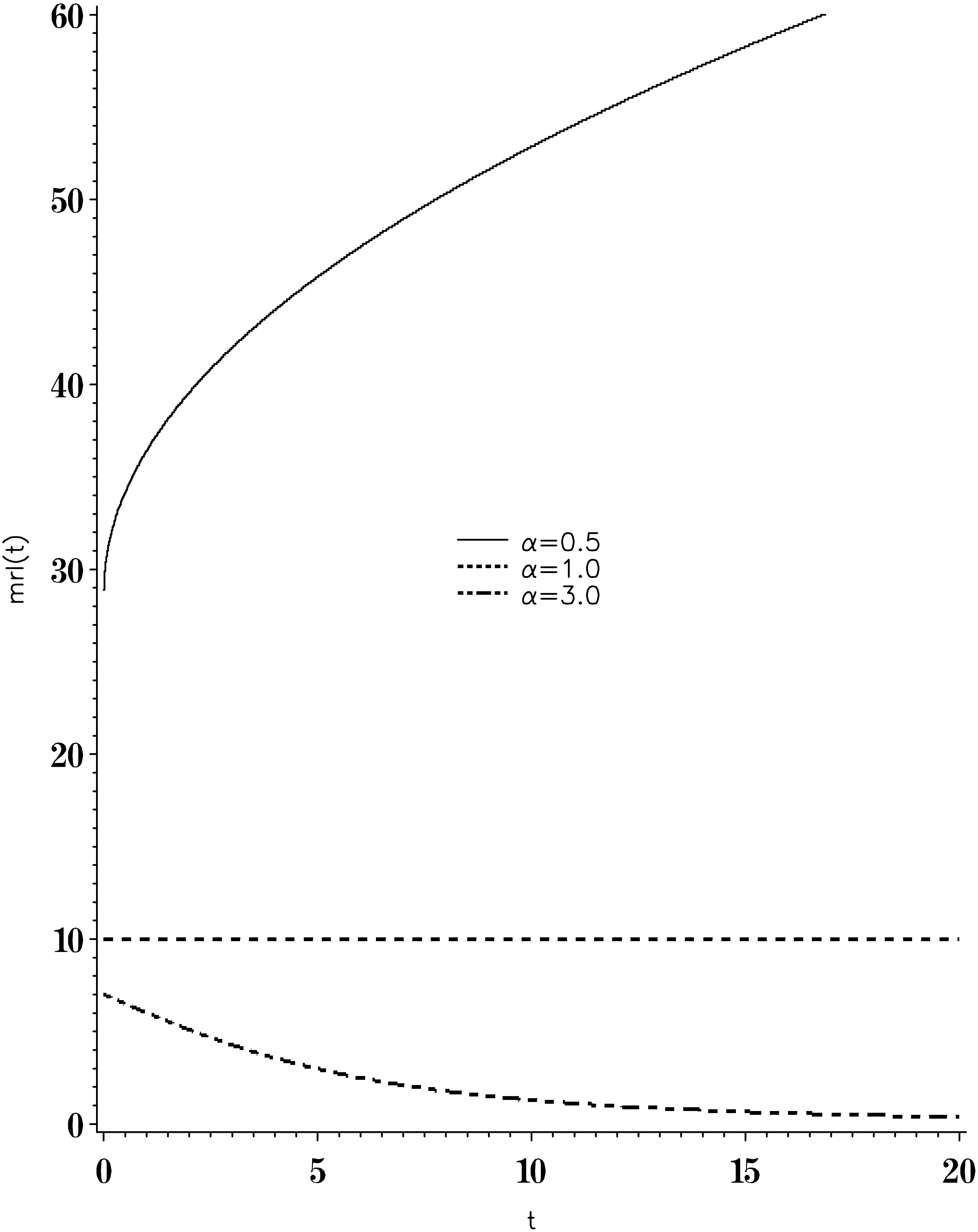

Figure 4
Comparison of Weibull Median Residual Life Functions