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Abstract

Recently Rowe and Logan (2004) introduced a complex fMRI activation model in which multiple regressors were allowed, hypothesis tests were formulated in terms of contrasts, and the phase was directly modeled as a fixed unknown quantity which may be estimated voxel by voxel. This model was shown to achieve higher detection power over the usual magnitude-only normal model especially at decreased signal-to-noise ratios. Here, we extend this model to allow for a dynamic rather than constant phase. It is seen that this dynamic phase complex fMRI model has identical regression coefficients and activation F-statistics as that of the magnitude-only model although derived with the phase included. It is also seen that the maximum likelihood estimate of the variance in this model is not consistent.

1 Introduction

It is well known that due to phase imperfections, fMRI voxel time course measurements appear in both the real and imaginary channels [2, 5, 6]. Recently Rowe and Logan (2004)

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introduced a complex fMRI activation model [9] in which multiple regressors were allowed, hypothesis tests were formulated in terms of contrasts, and the phase was directly modeled as a fixed unknown quantity [7] which may be estimated voxel by voxel. This model was shown to achieve higher detection power over the usual magnitude-only normal model [1, 3] especially at a decreased signal-to-noise ratio (SNR). Here, we extend this model to allow for a dynamic rather than constant phase.

Task related magnitude-only activation maps can be generated from complex valued voxel time courses that account for temporal changes in the phase. We will show that inference on task-related activation is equivalent between the dynamic phase complex fMRI model and the magnitude-only model in terms of having identical regression coefficients and likelihood ratio F-statistics although derived with the phase included. However, a detailed examination shows that the maximum likelihood estimate of the variance in the dynamic phase model is inconsistent, because the number of parameters increases with the sample size. An unbiased estimate can be obtained which is identical to the unbiased variance estimate from the magnitude-only model. Therefore the magnitude-only model results can be directly derived from a complex data model which allows for a dynamic phase.

2 Model

The complex fMRI activation model of Rowe and Logan (2004) can be written more generally as

$$\underset{2n\times 1}{y} = \underset{2n\times 2n}{\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}} \underset{2n\times 2(q+1)}{\begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix}} \underset{2(q+1)\times 1}{\begin{pmatrix} \beta \\ \beta \end{pmatrix}} + \underset{2n\times 1}{\eta} \tag{2.1}$$

where the observed vector of data $y = (y_R', y_I')'$ is the vector of observed real values stacked on the vector of observed imaginary values and the vector of errors $\eta = (\eta_R', \eta_I')' \sim \mathcal{N}(0, \Sigma \otimes \Phi)$ is similarly defined. Here we specify that $\Sigma = \sigma^2 I_2$ and $\Phi = I_n$. Further, A_1 and A_2 are are square diagonal matrices with t^{th} diagonal element $\cos\theta_t$ and $\sin\theta_t$, respectively. Note that if $\theta_t = \theta$ for all t, then $A_1 = \cos\theta I_n$, $A_2 = \sin\theta I_n$, and this becomes the constant phase complex model proposed by Rowe and Logan (2004). If there is a single constant column in X, then it can be shown that this reduces to a constant magnitude and different phase temporal fMRI model that is analogous to a previously presented constant magnitude different phase spatial MRI model [10].

This model generalization allows for dynamic temporal changes in the phase. This implies that one can test the hypotheses regarding task related changes in the magnitude of the complex voxel time courses while accounting for dynamic temporal changes in the phase, expressed as $H_0 : C\beta = 0$. For example, with a model with β_0 representing an intercept, β_1 representing a linear drift over time, and β_2 representing a contrast effect of a stimulus. Then to test whether the coefficient for the reference function or stimulus is 0, set $C = (0, 0, 1)$, so that the hypothesis is $H_0 : \beta_2 = 0$.

2.1 Parameter Estimates

As with the usual magnitude-only normal regression model and the constant phase complex nonlinear multiple regression model, we can obtain unrestricted maximum likelihood estimates of the parameters as derived in the appendix to be

$$\begin{aligned} \hat{\theta}_t &= \tan^{-1}\left(\frac{y_{It}}{y_{Rt}}\right), \quad t = 1, \ldots, n \\ \hat{\beta} &= (X'X)^{-1}X'\left(\hat{A}_1 y_R + \hat{A}_2 y_I\right), \\ \hat{\sigma}^2 &= \frac{1}{2n}\left[y - \begin{pmatrix} \hat{A}_1 X\hat{\beta} \\ \hat{A}_2 X\hat{\beta} \end{pmatrix}\right]'\left[y - \begin{pmatrix} \hat{A}_1 X\hat{\beta} \\ \hat{A}_2 X\hat{\beta} \end{pmatrix}\right], \end{aligned} \tag{2.2}$$

where $\hat{A}_1$ and $\hat{A}_2$ are diagonal matrices with $\cos\hat{\theta}_t$ and $\sin\hat{\theta}_t$ as the t^{th} diagonal element. Note that the estimate of the regression coefficients is a temporally "weighted" linear combination of estimates from the real and imaginary parts.

The estimated regression coefficients for the dynamic phase complex activation model can be shown to be equivalent to the usual magnitude-only ones as follows

$$\begin{aligned} \hat{\beta} &= (X'X)^{-1}X'\left(\hat{A}_1 y_R + \hat{A}_2 y_I\right) \\ &= (X'X)^{-1}X' vec\left(\frac{y_{Rt}}{\sqrt{y_{Rt}^2 + y_{It}^2}} y_{Rt} + \frac{y_{It}}{\sqrt{y_{Rt}^2 + y_{It}^2}} y_{It}\right) \\ &= (X'X)^{-1}X' y_M \end{aligned} \tag{2.3}$$

where here $vec(\cdot)$ is used to denote an n dimensional vector whose t^{th} element is given by its scalar argument and $y_M = vec\left(\sqrt{y_{Rt}^2 + y_{It}^2}\right)$.

The maximum likelihood estimates under the constrained null hypothesis $H_0 : C\beta = 0$ are similarly derived in the appendix and given by

$$
\begin{aligned}
\tilde{\theta}_t &= \tan^{-1}\left(\frac{y_{It}}{y_{Rt}}\right), \qquad t = 1, \dots, n \\
\tilde{\beta} &= \Psi\hat{\beta}, \\
\tilde{\sigma}^2 &= \frac{1}{2n}\left[y - \begin{pmatrix} \tilde{A}_1 X\tilde{\beta} \\ \tilde{A}_2 X\tilde{\beta} \end{pmatrix}\right]' \left[y - \begin{pmatrix} \tilde{A}_1 X\tilde{\beta} \\ \tilde{A}_2 X\tilde{\beta} \end{pmatrix}\right] \\
\Psi &= I_{q+1} - (X'X)^{-1}C'[C(X'X)^{-1}C']^{-1}C \ , \qquad (2.4)
\end{aligned}
$$

where $\tilde{A}_1$ and $\tilde{A}_2$ are diagonal matrices with $\cos\tilde{\theta}_t$ and $\sin\tilde{\theta}_t$ as the t^{th} diagonal element. The restricted regression coefficients can also be shown to be equivalent to the magnitude-only model because the multiplicative factor Ψ is identical in both cases.

2.2 Activation Statistics

The likelihood ratio statistic in Equation A.3 with some algebra can be written as

$$
F = \frac{(n-q-1)}{r}\left(\lambda^{-1/n} - 1\right) = \frac{(n-q-1)}{r}\frac{\hat{\beta}'C'[C(X'X)^{-1}C']^{-1}C\hat{\beta}}{2n\hat{\sigma}^2} \ . \qquad (2.5)
$$

Note that since

$$
\begin{aligned}
2n\hat{\sigma}^2 &= \left[y - \begin{pmatrix} \hat{A}_1 X\hat{\beta} \\ \hat{A}_2 X\hat{\beta} \end{pmatrix}\right]' \left[y - \begin{pmatrix} \hat{A}_1 X\hat{\beta} \\ \hat{A}_2 X\hat{\beta} \end{pmatrix}\right] \\
&= \sum_{t=1}^{n}\left[y_{Rt}^2 - 2y_{Rt}(x_t'\hat{\beta})\cos\hat{\theta}_t + \hat{\beta}x_t x_t'\hat{\beta}\cos^2\hat{\theta}_t + y_{It}^2 - 2y_{It}(x_t'\hat{\beta})\sin\hat{\theta}_t + \hat{\beta}x_t x_t'\hat{\beta}\sin^2\hat{\theta}_t\right] \\
&= \sum_{t=1}^{n}[y_{Mt} - x_t'\hat{\beta}]^2 \qquad (2.6)
\end{aligned}
$$

equals the error sum of squares from the magnitude-only model, the F statistic and equivalent likelihood ratio statistic is identical to the one from the magnitude-only model. In either case the F statistic follows the same distribution. If the signal-to-noise ratio is large so that y_{Mt} is approximately normal, then F follows an $F_{r,n-q-1}$ distribution under the null hypothesis, where r is the full row rank of C. Otherwise, one might use the Ricean distribution [4, 8] to derive the proper distribution of the F statistic. In either case, the estimates of β and the likelihood ratio test depend only on the magnitude data.

Note from (2.6) that the maximum likelihood estimate of σ^2 from the dynamic phase complex model is inconsistent, since it can be shown as follows that its expected value does not converge in probability or tend to its populaton value as the sample size tends to infinity

$$\begin{aligned} E\left(\hat{\sigma^2}\right) &= \frac{1}{2n} E\left\{\sum_{t=1}^{n} [y_{Mt} - x_t'\hat{\beta}]^2\right\} \\ &= \frac{1}{2n}\left\{(n-q-1)\sigma^2\right\} \\ &\xrightarrow{p} \frac{\sigma^2}{2}. \end{aligned}$$

An unbiased estimate of the variance can be obtained by simply using the unbiased estimate of the variance from the magnitude-only model.

3 Conclusions

A generalization of the constant phase complex activation fMRI model of Rowe and Logan (2004) was developed, where the phase angle is allowed to vary at each time point. It is shown that the estimated regression coefficients and the likelihood ratio F statistic for this dynamic phase complex fMRI model are equivalent to those in the usual magnitude-only model. It is also seen that the maximum likelihood estimate of the variance in this model is not consistent, but that a consistent variance estimate is obtained by simply using the magnitude-only unbiased variance estimate. Therefore, inference on task-related magnitude activation which is equivalent to that of the magnitude-only model can be derived directly from the dynamic phase complex model.

A Generalized Likelihood Ratio Test

A.1 Complex Model with θ_t

In applications using multiple regression including fMRI, we often wish to test linear contrast hypothesis (for each voxel) such as

$$
\begin{array}{lcclcl}
H_0: \ C\beta & = & \gamma \quad vs & H_1: \ C\beta & \neq & \gamma \\
\theta_t & \neq \theta_{t'} & & \theta_t & \neq \theta_{t'} & \\
\sigma^2 & > & 0 & \sigma^2 & > & 0 \ ,
\end{array}
$$

where C is an $r \times (q+1)$ matrix of full row rank and γ is an $r \times 1$ vector.

The likelihood ratio statistic is computed by maximizing the likelihood $p(y|\beta, \theta, \sigma^2, X)$ with respect to β, θ, and σ^2 under the null and alternative hypotheses where $\theta' = (\theta_1, \ldots, \theta_n)$. Denote the maximized values under the null hypothesis by $(\tilde{\beta}, \tilde{\theta}, \tilde{\sigma}^2)$ and those under the alternative hypothesis as $(\hat{\beta}, \hat{\theta}, \hat{\sigma}^2)$. These maximized values are then substituted into the likelihoods and the ratio taken. With the aforementioned distributional specifications, the likelihood of the model is

$$
p(y|X, \beta, \theta, \sigma^2) = (2\pi\sigma^2)^{-\frac{2n}{2}} e^{-\frac{h}{2\sigma^2}} \tag{A.1}
$$

where

$$
\begin{aligned}
h &= \left[y - \begin{pmatrix} A_1 X\beta \\ A_2 X\beta \end{pmatrix} \right]' \left[y - \begin{pmatrix} A_1 X\beta \\ A_2 X\beta \end{pmatrix} \right] \\
&= \beta'(X'X)\beta - 2\beta' X' [A_1' y_R + A_2' y_I] + y'y
\end{aligned}
$$

The logarithm of this likelihood can be writen as

$$
\begin{aligned}
LL &= -n\log(2\pi) - n\log\sigma^2 - \frac{1}{2\sigma^2}\beta'(X'X)\beta - \frac{1}{2\sigma^2}y'y \\
&\quad + \frac{1}{\sigma^2}\sum_{t=1}^{n} y_{Rt}\cos\theta_t x_t'\beta + \frac{1}{\sigma^2}\sum_{t=1}^{n} y_{It}\sin\theta_t x_t'\beta
\end{aligned} \tag{A.2}
$$

that we will use for maximization. Under the null hypothesis, the term $\psi'(C\beta - \gamma)/2$ needs to be added to the logarithm of the likelihood for the Lagrange multiplier constraint.

Unrestricted MLE's

Maximizing this likelihood with respect to the parameters is the same as maximizing the logarithm of the likelihood with respect to the parameters and yields

$$\begin{aligned}
\left.\frac{\partial LL}{\partial \beta}\right|_{\beta=\hat\beta,\theta=\hat\theta,\sigma^2=\hat\sigma^2} &= -\frac{1}{2\hat\sigma^2}\left[2(X'X)\hat\beta - 2X'\left(\hat A_1 y_R + \hat A_2 y_I\right)\right] \\
\left.\frac{\partial LL}{\partial \theta_t}\right|_{\beta=\hat\beta,\theta=\hat\theta,\sigma^2=\hat\sigma^2} &= -\frac{1}{\hat\sigma^2}\left[y_{Rt}x_t'\hat\beta(-1)\sin\hat\theta_t + y_{It}x_t'\hat\beta\cos\hat\theta_t\right] \qquad t=1,\ldots,n \\
\left.\frac{\partial LL}{\partial \sigma^2}\right|_{\beta=\hat\beta,\theta=\hat\theta,\sigma^2=\hat\sigma^2} & \quad -\frac{2n}{2}\frac{1}{\hat\sigma^2} + \frac{\hat h}{2}\frac{1}{(\hat\sigma^2)^2}
\end{aligned}$$

where $\hat h$ is h with MLE's substituted in. By setting these derivatives equal to zero and solving, we get the MLE's under the unrestricted model given in Equation 2.2.

Restricted MLE's

Maximizing this likelihood with respect to the parameters is the same as maximizing the logarithm of the likelihood in Equation A.2 with respect to the parameters with the Lagrange multiplier term $\psi'(C\beta-\gamma)/2$ added for the alternative hypothesis restriction and yields

$$\begin{aligned}
\left.\frac{\partial LL}{\partial \beta}\right|_{\beta=\tilde\beta,\theta=\tilde\theta,\psi=\tilde\psi,\sigma^2=\tilde\sigma^2} &= -\frac{1}{2\tilde\sigma^2}\left[2(X'X)\hat\beta - 2X'\left(\tilde A_1 y_R + \tilde A_2 y_I\right)\right] + \frac{1}{2}C'\tilde\psi \\
\left.\frac{\partial LL}{\partial \theta_t}\right|_{\beta=\tilde\beta,\theta=\tilde\theta,\psi=\tilde\psi,\sigma^2=\tilde\sigma^2} &= -\frac{1}{\tilde\sigma^2}\left[y_{Rt}x_t'\tilde\beta(-1)\sin\tilde\theta_t + y_{It}x_t'\tilde\beta\cos\tilde\theta_t\right] \qquad t=1,\ldots,n \\
\left.\frac{\partial LL}{\partial \psi}\right|_{\beta=\tilde\beta,\theta=\tilde\theta,\psi=\tilde\psi,\sigma^2=\tilde\sigma^2} &= \frac{1}{2}(C\tilde\beta-\gamma) \\
\left.\frac{\partial LL}{\partial \sigma^2}\right|_{\beta=\tilde\beta,\theta=\tilde\theta,\psi=\tilde\psi,\sigma^2=\tilde\sigma^2} &= -\frac{2n}{2}\frac{1}{\tilde\sigma^2} + \frac{\tilde h}{2}\frac{1}{(\tilde\sigma^2)^2}
\end{aligned}$$

where $\tilde h$ is h with MLE's substituted in. By setting these derivatives equal to zero and solving, we get the MLE's under the restricted model given in Equation 2.4.

Note that $\hat\sigma^2 = \hat h/(2n)$ and $\tilde\sigma^2 = \tilde h/(2n)$. Then the generalized likelihood ratio is

$$\lambda = \frac{p(y|\tilde\beta,\tilde\sigma^2,\tilde\theta,X)}{p(y|\hat\beta,\hat\sigma^2,\hat\theta,X)} = \frac{\left(\tilde\sigma^2\right)^{-2n/2} e^{-2\tilde h n/(2\tilde h)}}{\left(\hat\sigma^2\right)^{-2n/2} e^{-2\hat h n/(2\hat h)}}\ , \tag{A.3}$$

and Equation 2.5 follows.

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